Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

Project I. Existence of finite atlas on a manifold

Let $M \subseteq \mathbb{R}^n$ be a nonempty d-dimensional C^k-manifold, where $d \geq 1$ and $k \in \mathbb{N}_\infty^+$. Recall that a C^k-chart on M is a pair (U, φ), where U is open in M and φ is a C^k-diffeomorphism from U onto an open subset of $\mathbb{R}^d$. An **atlas** is a family of such charts whose domains cover M. Neither a chart domain nor its image is required to be connected.

The purpose of this project is to construct a finite atlas by studying open coverings. We first obtain a bound of 2^n charts when M is compact, then use compact exhaustion to obtain a bound of $3 \cdot 2^n$ charts without assuming compactness.

An open cover $\mathcal{V}$ of a space X is a **refinement** of an open cover $\mathcal{U}$ if each member of $\mathcal{V}$ is contained in a member of $\mathcal{U}$. A cover has **multiplicity at most** q if each point belongs to at most q of its members; its **order** is then at most $q - 1$. For a nonempty metric space X, its **Lebesgue covering dimension** is the least integer $m \geq 0$ such that every open cover has an open refinement of multiplicity at most $m + 1$, or ∞ if no such integer exists.

I.1. Let X be a compact metric space and $\mathcal{U}$ an open cover of X. Show that there exists $\delta > 0$ such that every nonempty subset of X of diameter less than δ is contained in a member of $\mathcal{U}$. Such a δ is called a **Lebesgue number** of the cover.

Hint: for each $x \in X$, choose $r_x > 0$ such that $B_X(x, 2r_x)$ is contained in a member of the cover. Extract a finite subcover from $\{B_X(x, r_x) \mid x \in X\}$ and take the minimum of the corresponding radii.

I.2. Fix $h > 0$. Consider the grid of closed boxes

$$Q_a = \prod_{i=1}^{n} [ha_i, h(a_i + 1)], \qquad a = (a_1, \ldots, a_n) \in \mathbb{Z}^n.$$

For each vertex ha, let S_a be the interior in $\mathbb{R}^n$ of the union of all grid boxes containing ha.

(1) Show that

$$S_a = \prod_{i=1}^{n} (h(a_i - 1), h(a_i + 1)).$$

In particular, $\operatorname{diam}(S_a) = 2h\sqrt{n}$.

(2) Show that the sets S_a cover $\mathbb{R}^n$, and that each point belongs to at most 2^n of them.

(3) Show that only finitely many of the sets S_a meet any given bounded subset of $\mathbb{R}^n$.

I.3. Let $X \subseteq \mathbb{R}^n$ be nonempty and compact, and let $\mathcal{U}$ be an open cover of X in its subspace topology. Choose a Lebesgue number δ for this cover and take $h > 0$ with $2h\sqrt{n} < \delta$. Show that the nonempty sets

$$V_a = X \cap S_a$$

form a finite open refinement of $\mathcal{U}$ with multiplicity at most 2^n. Deduce that the Lebesgue covering dimension of X, and in particular of a compact M, is at most $2^n - 1$.

Notice that the vertices need not belong to X, and containment in a member of $\mathcal{U}$ is asserted for V_a, not for the whole set S_a.

I.4. Assign to the vertex ha the color

$$(a_1 \bmod 2, \dots, a_n \bmod 2) \in \{0,1\}^n.$$

Show that $S_a \cap S_b = \emptyset$ whenever $a \neq b$ have the same color. Thus the refinement in Exercise I.3 is the union of at most 2^n families, each consisting of pairwise disjoint open sets.

This disjointness is the additional property that we will use to combine chart domains.

I.5. Let (W_j, ψ_j), $j \in J \subseteq \mathbb{N}^+$, be a finite or countable family of C^k-charts with pairwise disjoint domains on a manifold of dimension $d \geq 1$. Show that $W = \bigcup_{j \in J} W_j$ is the domain of a single C^k-chart.

Hint: the map

$$T : \mathbb{R}^d \to B(0,1), \qquad T(y) = \frac{y}{\sqrt{1 + \|y\|^2}},$$

is a smooth diffeomorphism. Let $e_1 = (1, 0, \dots, 0)$ and define

$$\psi(x) = 3j e_1 + T(\psi_j(x)), \qquad x \in W_j.$$

Verify that its image is open, that it is bijective onto its image, and that both ψ and its inverse are of class C^k.

I.6. Assume that M is compact. Apply the preceding construction to its cover by the domains of any C^k-atlas. Restrict an appropriate chart to each nonempty $M \cap S_a$, then combine the resulting charts of each color. Conclude that M admits a C^k-atlas consisting of at most 2^n charts.

I.7. Without assuming compactness, show that M has a **compact exhaustion**: compact subsets $K_j \subseteq M$, $j \geq 1$, such that

$$K_j \subseteq \operatorname{Int}_M K_{j+1}, \qquad M = \bigcup_{j \geq 1} K_j.$$

All interiors and closures here are taken in M.

Hint: use charts and the countable basis inherited from $\mathbb{R}^n$ to obtain a countable open cover $(U_i)_{i \geq 1}$ with each $\overline{U_i}$ compact. Starting with $K_1 = \overline{U_1}$, cover K_j by finitely many of the U_i, and let K_{j+1} be the union of their closures and $\overline{U_1}, \dots, \overline{U_{j+1}}$. The sets $M \cap \overline{B(0,j)}$ need not be compact, since M need not be closed in $\mathbb{R}^n$.

I.8. Choose a compact exhaustion as above, put $K_{-1} = K_0 = \emptyset$, and define

$$O_j = \operatorname{Int}_M K_{j+1} \setminus K_{j-2}, \qquad j \geq 1.$$

(1) Show that the O_j are open in M and cover M.

Hint: for $x \in M$, take the least $j \geq 1$ such that $x \in K_j$.

(2) Show that $O_j \cap O_l = \emptyset$ if $|j - l| \geq 3$. Thus layers whose indices have the same residue modulo 3 are pairwise disjoint.

(3) Show that every point has a neighborhood meeting only finitely many layers.

I.9. Let $\mathcal{U}$ be any open cover of M.

(1) For each j, restrict $\mathcal{U}$ to the compact set K_{j+1} and choose a sufficiently small grid size $h_j > 0$. Writing $S_{j,a}$ for its vertex stars, show that the nonempty sets

$$V_{j,a} = O_j \cap S_{j,a}, \qquad a \in \mathbb{Z}^n,$$

form a finite open cover of O_j by sets open in M, each contained in a member of $\mathcal{U}$.

Hint: first apply Exercise I.3 to K_{j+1}, then intersect its refinement with O_j. The compact set K_{j+1} need not itself be a manifold.

(2) Give $V_{j,a}$ the color

$$(j \bmod 3, a_1 \bmod 2, \ldots, a_n \bmod 2).$$

Show that the sets of any one color are pairwise disjoint. Deduce that every open cover of M has a countable open refinement of multiplicity at most $3 \cdot 2^n$, and hence that its Lebesgue covering dimension is finite.

(3) Apply this refinement to a cover by chart domains. Use Exercise I.5 to combine the charts of each color, and conclude that M admits a C^k-atlas with at most $3 \cdot 2^n$ charts.

Further fact (no proof requested). A manifold of dimension d has Lebesgue covering dimension d; see Munkres, [2, Chapter 8, § 50]. Using simplicial complexes instead of grid boxes gives refinements of chart covers that split into $d + 1$ families of pairwise disjoint open sets. The same chart-gluing argument therefore gives a C^k-atlas with at most $d+1$ charts, even when M is noncompact. See Hurewicz–Wallman, [1], for more on dimension theory. As above, chart domains may be disconnected and $d \geq 1$. For $d = 0$, each nonempty chart domain is a singleton: a compact manifold has a finite atlas, whereas an infinite discrete manifold does not.

References

[1] Witold Hurewicz and Henry Wallman, *Dimension theory*, Princeton University Press, Princeton, 1948 (English).

[2] James R. Munkres, *Topology.*, 2nd ed., Upper Saddle River, NJ: Prentice Hall, 2000 (English).