Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

Project IV. Existence of many self-diffeomorphisms

The goal of this problem set is to establish the following theorem, which confirms our geometric intuition that there should be many self-diffeomorphisms on a smooth manifold and that we may even obtain them by smoothly deforming the identity. The approach given here follows closely the treatment in [1, § I.2].

Theorem. If M is a connected smooth manifold with $\dim M \geq 2$, and if $p_1, \ldots, p_k$ are finitely many distinct points in M, then for any distinct points $q_1, \ldots, q_k$ in M, there exists a diffeomorphism $\varphi : M \to M$ that is homotopic to Id_M such that $\varphi(p_i) = q_i$ for each $i = 1, \ldots, k$.

In the language of group actions, this implies in particular that the natural group action $\mathrm{Diff}(M) \times M \to M$, $(\psi, x) \mapsto \psi(x)$ is highly transitive for any connected smooth manifold M of dimension at least two, where $\mathrm{Diff}(M)$ is the group of self-diffeomorphisms. Do not worry if you are not yet familiar with group actions or the meaning of high transitivity: this terminology merely restates part of the theorem above.

Let M and N be smooth manifolds and let φ and ψ be smooth maps from M to N. We say that φ is homotopic to ψ, and write $\varphi \sim \psi$, if there exists a smooth map

$$H : \mathbb{R} \times M \to N$$

such that

$$H(0, x) = \varphi(x) \qquad \text{and} \qquad H(1, x) = \psi(x), \qquad x \in M.$$

IV.1. Show that for all smooth maps φ and ψ from M to N, we have $\varphi \sim \varphi$, and if $\varphi \sim \psi$, then $\psi \sim \varphi$.

IV.2. Construct a smooth non-decreasing function $g \in C^\infty(\mathbb{R})$ such that $g(t) = 0$ for $t \leq 0$ and $g(t) = 1$ for $t \geq \frac{1}{4}$.

IV.3. Show that for all smooth maps φ_1, φ_2 and φ_3 from M to N, if $\varphi_1 \sim \varphi_2$ and $\varphi_2 \sim \varphi_3$, then $\varphi_1 \sim \varphi_3$.

IV.4. Conclude that the homotopy relation $\sim$ is an equivalence relation on the set of all smooth maps from M to N.

Hint: let g be the function in Exercise IV.2, and let H (resp. K) be a homotopy from φ_1 (resp. φ_2) to φ_2 (resp. φ_3), and consider

$$L(t, x) = \begin{cases} H(g(t), x), & t \leq \dfrac{1}{2} \\ K\left(g\left(t - \dfrac{3}{4}\right), x\right), & t > \dfrac{1}{2}. \end{cases}$$

IV.5. Suppose M, N, and Q are smooth manifolds, and $\varphi, \psi : M \to N$ and $\varphi_1, \psi_1 : N \to Q$ are smooth maps. Show that if $\varphi \sim \psi$ via a homotopy H and $\varphi_1 \sim \psi_1$ via a homotopy G, then $\varphi_1 \circ \varphi \sim \psi_1 \circ \psi$.

Hint: consider the map

$$K(t, x) = G(t, H(t, x)).$$

IV.6. Show that $\varphi, \psi : M \to N$ are homotopic if and only if there is a smooth map $K : \mathbb{R} \times M \to N$ such that

$$K(t,x) = \varphi(x), \ t \leq 0 \qquad \text{and} \qquad K(t,x) = \psi(x), \ t \geq 1.$$

IV.7. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} \exp\left(-\dfrac{t^2}{9-t^2}\right), & |t| < 3; \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f \in C^\infty(\mathbb{R})$, $\operatorname{supp}(f) = [-3,3]$ and $|f'| < 1$.

Hint: to prove $|f'| < 1$, calculate $[f'(t)]^2$ for $|t| < 3$, and then make the change of variable $t = 3\sin\theta$, $|\theta| < \pi/2$.

IV.8. Show that there exists a diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that

(1) $\varphi(0,\ldots,0) = (1,0,\ldots,0)$;

(2) $\varphi(x) = x$ for every $x \in \mathbb{R}^n$ such that $\|x\|_\infty > 3$.

Hint: consider

$$\varphi(\xi_1,\ldots,\xi_n) = \left(\xi_1 + \prod_{i=1}^{n} f(\xi_i), \xi_2, \ldots, \xi_n\right),$$

where f is the function in Exercise IV.7.

IV.9. Suppose C is a closed subset of M such that $M \setminus C$ is connected. Let $\sim$ be the equivalence relation on $M \setminus C$ defined as follows:

$x_1 \sim x_2$ if and only if there is a diffeomorphism $\varphi : M \to M$ homotopic to Id_M such that $\varphi(x_1) = x_2$ and $\varphi|_C = \mathrm{Id}_C$.

Show that all points in $M \setminus C$ are equivalent.

Hint: use Exercise IV.8 to show that each equivalence class is open in $M \setminus C$.

IV.10. Complete the proof of the theorem by induction on k.

Hint: first show that removing finitely many points from a connected manifold of dimension at least two leaves it connected, using the connectedness of punctured coordinate balls. Then use Exercises IV.6 and IV.9.

References

[1] Werner Greub, Stephen Halperin, and Ray Vanstone, *Connections, curvature, and cohomology. Vol. I: De Rham cohomology of manifolds and vector bundles*, Pure Appl. Math., Academic Press, vol. 47, Academic Press, New York, NY, 1972 (English).