Lecture Notes on
Mathematical Analysis III

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Contents

CHAPTER 1

Smooth manifolds

Throughout this chapter, n denotes a positive integer, and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, unless explicitly stated otherwise. Correspondingly, by functions, we mean functions taking values in $\mathbb{K}$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, the symbol $|x|$ denotes its Euclidean norm $\sqrt{x_1^2 + \cdots + x_n^2}$. For $a \in \mathbb{R}^n$ and $r > 0$, the symbol $B(a, r)$ denotes the *open* ball of center a and radius r, where the metric is the one induced by the norm $|\cdot|$. We denote by $\mathbb{N}$ the set of all natural numbers, including zero. The set of all positive integers is denoted by $\mathbb{N}^+$. We also set $\mathbb{N}^+_\infty = \mathbb{N}^+ \cup \{\infty\}$ and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$.

1. Continuously differentiable functions of higher order

Definition 1.1. Let Ω be an open set in $\mathbb{R}^n$. We define the class $C^k(\Omega)$ of **continuously differentiable functions of order** $k \in \mathbb{N}_\infty$ inductively as follows:

(1) $C^0(\Omega)$ is the space of all continuous functions;
(2) if $k \in \mathbb{N}^+$, the space $C^k(\Omega)$ consists of all continuously differentiable functions f such that $\partial_i f \in C^{k-1}(\Omega)$ for all $1 \leq i \leq n$;
(3) the space $C^\infty(\Omega)$ is the intersection of all $C^k(\Omega)$, $k \in \mathbb{N}$.

Functions in $C^k(\Omega)$ are also called **of class** C^k, and functions in $C^\infty(\Omega)$ are also called **smooth**.

Proposition 1.2. *Let Ω be an open set in $\mathbb{R}^n$ and $k \in \mathbb{N}_\infty$. The following hold:*

(1) *With respect to the usual pointwise addition of functions and multiplication by scalars, $C^k(\Omega)$ is a vector space over $\mathbb{K}$.*
(2) *Under pointwise multiplication and the above vector space structure, $C^k(\Omega)$ becomes a commutative algebra over $\mathbb{K}$, i.e. in addition to the usual axioms for a $\mathbb{K}$-vector space, with 1_Ω denoting the constant function on Ω with value $1 \in \mathbb{K}$, the following conditions are satisfied for all $f, g, h \in C^k(\Omega)$:*
   (2.a) $fg = gf$;
   (2.b) $(fg)h = f(gh)$;
   (2.c) $f(g + h) = fg + fh$ *and* $(f + g)h = fh + gh$;
   (2.d) $(cf)g = c(fg) = f(cg)$ *for all* $c \in \mathbb{K}$;
   (2.e) $1f = f1 = f$.

Proof. Exercise. □

Definition 1.3. Let $\Omega \subseteq \mathbb{R}^n$ be open, $f = (f_1, \dots, f_m) : \Omega \to \mathbb{R}^m$, where $m, n \in \mathbb{N}^+$. We say f is **continuously differentiable of higher order** $k \in \mathbb{N}_\infty$, or of class C^k, if for each $1 \le i \le m$, we have $f_i \in C^k(\Omega)$. We denote by $C^k(\Omega, \mathbb{R}^m)$ the $\mathbb{R}$-vector space of all maps from Ω to $\mathbb{R}^m$ that are of class C^k.

Proposition 1.4. *Let $l, m, n \in \mathbb{N}^+$ and $k \in \mathbb{N}_\infty$. Suppose $U \subseteq \mathbb{R}^l$ and $V \subseteq \mathbb{R}^m$ are open and $f : U \to V \subseteq \mathbb{R}^m$ and $g : V \to \mathbb{R}^n$ are of class C^k. Then $g \circ f \in C^k(U, \mathbb{R}^n)$.*

Proof. Exercise. □

Exercises

1.1. Prove Proposition 1.2.
1.2. Prove Proposition 1.4.
1.3. (**Polynomials and the multi-index notation**) Let $n \in \mathbb{N}^+$. A **monomial function** on $\mathbb{R}^n$ is a function of the form $(x_1, \dots, x_n) \mapsto x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where each $\alpha_i \in \mathbb{N}$ and we interpret x_i^0 as $1 \in \mathbb{K}$. To simplify notation, for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We also introduce the following notation:

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!, \qquad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} \qquad \text{and} \qquad |\alpha| = \sum_{i=1}^n \alpha_i.$$

A **polynomial function** on $\mathbb{R}^n$ is a function that is a linear combination of monomial functions.

(i) Show that every monomial function on $\mathbb{R}^n$ is smooth.
(ii) Show that every polynomial function on $\mathbb{R}^n$ is smooth.
(iii) Show that for every polynomial function f on $\mathbb{R}^n$ and every $a \in \mathbb{R}^n$, we have

$$\begin{aligned} \forall x \in \mathbb{R}^n, \qquad f(x) &= \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (\partial^\alpha f)(a)(x-a)^\alpha \\ &= f(a) + \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \frac{1}{\alpha!} (\partial^\alpha f)(a)(x-a)^\alpha, \end{aligned}$$

where the sums are essentially finite in the sense of $\partial^\alpha f \equiv 0$ when $|\alpha|$ is sufficiently large, so there is no problem of convergence in the above equalities.

Hint: first treat the one-variable case, then the monomial function case, and finally the polynomial function case.

1.4. (**Power series of several variables**) Let $a \in \mathbb{R}^n$ and $r \in \mathbb{R}^n_{>0}$. The **open polydisk** $\Delta(a, r)$ of center a and radii r is defined by

$$\Delta(a, r) := \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; |x_i - a_i| < r_i \text{ for each } 1 \le i \le n \right\}.$$

Recall the multi-index notation in the previous exercise, and let $c_\alpha \in \mathbb{K}$ for each $\alpha \in \mathbb{N}^n$.

(i) Show that $\Delta(a, r)$ is open in $\mathbb{R}^n$.

(ii) Suppose that there exists $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ such that $|t_i| = r_i$ for each $1 \le i \le n$, and that the series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha t^\alpha$$

converges unconditionally, i.e. it converges to the same limit in $\mathbb{K}$ with respect to any order of summation over the countable set $\mathbb{N}^n$. Show that there exists $M > 0$ such that

$$\forall \alpha \in \mathbb{N}^n, \qquad |c_\alpha| \le \frac{M}{r^\alpha} = \frac{M}{r_1^{\alpha_1} \cdots r_n^{\alpha_n}}.$$

(iii) Under the same assumption as in (ii), show that for any $x \in \Delta(a, r)$, the series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha (x - a)^\alpha$$

converges unconditionally to some $f(x) \in \mathbb{K}$.

(iv) Let $f : \Delta(a, r) \to \mathbb{K}$ be the function given in (iii) (so we keep the same assumption as in (ii)). Show that f is smooth, that

$$\forall \alpha \in \mathbb{N}^n, \qquad (\partial^\alpha f)(a) = \alpha! c_\alpha$$

and that

$$\forall x \in \Delta(a, r), \qquad (\partial^\alpha f)(x) = \sum_{\beta \ge \alpha} \frac{\beta!}{(\beta - \alpha)!} c_\beta (x - a)^{\beta - \alpha},$$

where the sum is over all $\beta \in \mathbb{N}^n$ such that $\beta_i \ge \alpha_i$ for each $1 \le i \le n$.

2. Smooth functions on arbitrary sets and partitions of unity

Definition 2.1. Let $A \subseteq \mathbb{R}^n$, $k \in \mathbb{N}_\infty$ and $f : A \to \mathbb{K}$. We say f is **of class** C^k at $a \in A$ if there exists an open neighborhood U_a of a and a function $g \in C^k(U_a)$ such that $f|_{A \cap U_a} = g|_{A \cap U_a}$. In the case $k = \infty$, we also say that f is **smooth at** a.

Definition 2.2. Let $A \subseteq \mathbb{R}^n$, $k \in \mathbb{N}_\infty$ and $f : A \to \mathbb{K}$. We say $f : A \to \mathbb{K}$ is **of class** C^k *on* A if there exists an open neighborhood U of A and a function $g \in C^k(U)$ such that $g|_A = f$. In the case $k = \infty$, we say that f is **smooth**.

Theorem 2.3. *If $f : A \to \mathbb{K}$ is a function on an arbitrary $A \subseteq \mathbb{R}^n$ and $k \in \mathbb{N}_\infty^+$, then f is of class C^k on A if and only if f is of class C^k at every $a \in A$.*

The proof of Theorem 2.3 requires some preparation. Of particular importance shall be the so-called *partitions of unity*, which serve as one of the most important tools in passing from “local” to “global”.

To properly formulate the notion of partitions of unity, we need to define the notion of the support of a function.

Definition 2.4. Let (X, d) be a metric space (or more generally let X be a topological space), and $f : X \to \mathbb{K}$ a function. The **support of** f, denoted by $\mathrm{supp}(f)$, is the complement of the largest open set in X on which f vanishes.

Proposition 2.5. *Let $f : X \longrightarrow \mathbb{K}$ be a function on a metric space (X, d). Show that the support of f exists and* $\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$. *Moreover,* $\operatorname{supp}(f)$ *is the smallest closed set containing* $\{x \in X \mid f(x) \neq 0\}$.

Proof. Exercise. $\square$

Remark 2.6. We shall often consider functions that are defined on a subspace Y of X, and we point out that the support of such a function $f : Y \longrightarrow \mathbb{K}$ is taken relative to the domain of f, i.e. Y, instead of the ambient space X. In view of Proposition 2.5, for such an f, we have $\operatorname{supp}(f) = \overline{\{y \in Y \mid f(y) \neq 0\}}$ where the closure is taken relative to Y, and it is the smallest closed subset relative to Y containing $\{y \in Y \mid f(y) \neq 0\}$, or dually, it is the complement in Y of the largest open set relative to Y on which f vanishes.

Definition 2.7. Let $M \subseteq \mathbb{R}^n$ (the notation M hints that we will mostly use the notion for manifolds) and suppose $\mathcal{U} = (U_\lambda)_{\lambda\in\Lambda}$ is an indexed family of open sets in $\mathbb{R}^n$ that covers M, i.e. $\bigcup_{\lambda\in\Lambda} U_\lambda \supseteq M$. A **partition of unity** on M subordinate to the open cover $\mathcal{U}$ is a family $(\varphi_\lambda)_{\lambda\in\Lambda}$ of *smooth* functions on $U := \bigcup_{\lambda\in\Lambda} U_\lambda$ indexed by the same Λ such that the following hold:

(1) (the subordination condition) for each $\lambda \in \Lambda$, we have $0 \leq \varphi_\lambda \leq 1$ (this means $0 \leq \varphi_\lambda(x) \leq 1$ for every $x \in U$) and
$$\operatorname{supp}(\varphi_\lambda) := \overline{\{x \in U \mid \varphi_\lambda(x) \neq 0\}} \subseteq U_\lambda,$$
where the closure is taken relative to U (see Remark 2.6).

(2) (the local finiteness condition) Let U be the union $\bigcup_{\lambda\in\Lambda} U_\lambda$. For each $x \in U$, there exists an open neighborhood W_x of x in U such that the set
$$\Lambda(W_x) := \left\{\lambda \in \Lambda \,\middle|\, \operatorname{supp}(\varphi_\lambda) \cap W_x \neq \emptyset\right\}$$
is finite. Hence $x \in U \mapsto \sum_{\lambda\in\Lambda} \varphi_\lambda(x)$ is a smooth function on U, because when restricted to W_x, it equals
$$\sum_{\lambda\in\Lambda(W_x)} \varphi_\lambda(y)$$
for each $y \in W_x$.

(3) (the unity condition) for each $x \in M$, we have $\sum_{\lambda\in\Lambda} \varphi_\lambda(x) = 1$.

Theorem 2.8 (Existence of partitions of unity). *Let M be an arbitrary subset of $\mathbb{R}^d$ with $d \in \mathbb{N}^+$. For any indexed family $\mathcal{U} = (U_\lambda)_{\lambda\in\Lambda}$ of open sets in $\mathbb{R}^d$ that covers M, there exists a partition of unity on M subordinate to $\mathcal{U}$.*

We will postpone the proof of Theorem 2.8. For now, we admit its validity and apply it to prove Theorem 2.3.

Proof of Theorem 2.3. By definition, if f is of class C^k on A, it is so at every $a \in A$. It remains to show the converse. Since f is of class C^k at every point in A, for each $a \in A$, there exists a pair (U_a, g_a), with U_a being an open neighborhood of a in $\mathbb{R}^n$

and $g_a \in C^k(U_a)$ such that $g_a|_{U_a\cap A} = f|_{U_a\cap A}$. Clearly, $\mathcal{U} := (U_a)_{a\in A}$ covers A. Hence by Theorem 2.8, there exists a partition of unity $(\varphi_a)_{a\in A}$ on A subordinate to $\mathcal{U}$. In particular, for each $x \in U := \bigcup_{a\in A} U_a$, there exists an open neighborhood W_x of x in U such that

$$A(W_x) := \left\{a \in A \,\middle|\, W_x \cap \operatorname{supp}(\varphi_a) \neq \emptyset\right\}$$

is finite.

For each $a \in A$, define $h_a : U \longrightarrow \mathbb{K}$ by

$$h_a(x) = \begin{cases} \varphi_a(x)g_a(x), & \text{if } x \in U_a \\ 0, & \text{otherwise.} \end{cases}$$

Set $V_a := U \setminus \operatorname{supp}(\varphi_a)$, which is open. Then $h_a|_{U_a} \in C^k(U_a)$ and $h_a|_{V_a} \equiv 0 \in C^\infty(V_a)$. Since $\operatorname{supp}\varphi_a \subseteq U_a$, we have $U_a \cup V_a = U$, and $h_a \in C^k(U)$. Define $h : U \longrightarrow \mathbb{K}$ by

$$h(x) = \sum_{a\in A} h_a(x).$$

For each $x_0 \in U$, choose an open neighborhood W_{x_0} of x_0 in U as in the previous paragraph. For any $x \in W_{x_0}$, by definition, we have $h_a(x) = 0$ unless $a \in A(W_{x_0})$. Hence from the finiteness of $A(W_{x_0})$, we have

$$h|_{W_{x_0}} = \sum_{a\in A(W_{x_0})} h_a|_{W_{x_0}} \in C^k(W_{x_0}). \tag{2.1}$$

Since $\bigcup_{x\in U} W_x = U$, by (2.1), h is a well-defined function of class C^k on U.

On the other hand, we claim that for any $x, a \in A$, we have $h_a(x) = \varphi_a(x)f(x)$. Indeed, if $x \in U_a$, we have

$$h_a(x) = \varphi_a(x)g_a(x) = \varphi_a(x)f(x);$$

otherwise $x \notin \operatorname{supp}\varphi_a$ since $\operatorname{supp}\varphi_a \subseteq U_a$, and $\varphi_a(x) = 0$, so we still have $h_a(x) = 0 = \varphi_a(x)f(x)$. This proves the claim. It now follows from the unity condition that

$$\forall x \in A, \qquad h(x) = \sum_{a\in A} h_a(x) = \sum_{a\in A} \varphi_a(x)f(x) = \left(\sum_{a\in A} \varphi_a(x)\right) f(x) = f(x),$$

so $h|_A = f$ with $h \in C^k(U)$, and f is of class C^k on A, as desired. □

In the remainder of this section, we give a proof of Theorem 2.8.

Lemma 2.9. *The function*

$$\begin{aligned} h : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto \begin{cases} e^{-1/t}, & t > 0 \\ 0 & t \leq 0 \end{cases} \end{aligned} \tag{2.2}$$

is non-decreasing, smooth and $0 \leq h \leq 1$.

PROOF. The only nontrivial part is the smoothness of h at 0. For this, we show by induction the claim that for any $n \in \mathbb{N}$, there exists a polynomial P_n over $\mathbb{R}$ such that $h^{(n)}(t) = P_n(1/t)h(t)$ for any $t > 0$. This clearly holds for $n = 0$ by taking $P_0 = 1$. If the claim holds for $n = k$, then for each $t > 0$, we have

$$h^{(k+1)}(t) = \left[h^{(k)}(t)\right]' = \left[P_k\left(\frac{1}{t}\right)\exp\left(-\frac{1}{t}\right)\right]' = \left[P_k'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right) + P_k\left(\frac{1}{t}\right)\frac{1}{t^2}\right]\exp\left(-\frac{1}{t}\right).$$

Hence the claim holds for $n = k+1$ with $P_{k+1}(x) = x^2[P_k(x) - P_k'(x)]$. As $t \to 0^+$, we have $1/t \to +\infty$. Since the exponential function grows faster than any polynomial function, we have $\lim_{t\to 0^+} h^{(n)}(t) = 0$ for each $n \in \mathbb{N}$, which implies the smoothness of h at 0. □

LEMMA 2.10. *Let h be the function in Lemma 2.9. For each $\varepsilon > 0$ and $a \in \mathbb{R}^n$, the function*

$$\begin{aligned} \varphi_{\varepsilon,a} : \mathbb{R}^n &\to \mathbb{R} \\ x &\mapsto h(\varepsilon^2 - |x-a|^2) \end{aligned} \tag{2.3}$$

is smooth, with $\varphi_{\varepsilon,a}(a) > 0$, $\varphi_{\varepsilon,a} \geq 0$ and $\operatorname{supp}\varphi_{\varepsilon,a} = \overline{B(a,\varepsilon)}$.

PROOF. This follows from a routine check, and we leave the details as an exercise (do it!). □

LEMMA 2.11. *The collection $\mathfrak{B} := \{B(a,r) \mid a \in \mathbb{Q}^n,\ r \in \mathbb{Q}_{>0}\}$ of open balls with rational centers and rational radii is a topological basis for $\mathbb{R}^n$ in the sense that any open set in $\mathbb{R}^n$ is a union of balls in $\mathfrak{B}$.*

PROOF. Let U be an open set in $\mathbb{R}^n$. Set $\mathfrak{B}(U) := \{B \in \mathfrak{B} \mid B \subseteq U\} \subseteq \mathfrak{B}$. Clearly, we have

$$U \supseteq \bigcup_{B\in\mathfrak{B}(U)} B. \tag{2.4}$$

Conversely, for any $x \in U$, since U is open, there exists an $\varepsilon > 0$ such that U contains the open ball $B(x,\varepsilon)$. In general, we have neither $x \in \mathbb{Q}^n$ nor $\varepsilon \in \mathbb{Q}$. But we may perturb x and ε slightly to obtain a ball $B(a,r)$ in $\mathfrak{B}$ with $x \in B(a,r) \subseteq U$. Indeed, since $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$, we may choose an $a \in \mathbb{Q}^n$ with $|x-a| < \frac{\varepsilon}{2}$. Similarly, the density of $\mathbb{Q}$ in $\mathbb{R}$ implies that we can find an $r \in \mathbb{Q}_{>0}$ such that $|x-a| < r < \frac{\varepsilon}{2}$. Now for any $y \in B(a,r)$, we have

$$|y-x| \leq |y-a| + |a-x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and $y \in B(x,\varepsilon) \subseteq U$. Hence

$$x \in B(a,r) \subseteq B(x,\varepsilon) \subseteq U, \qquad \text{and} \qquad B(a,r) \in \mathfrak{B}(U).$$

It follows that $U \subseteq \bigcup_{B\in\mathfrak{B}(U)} B$, which, combined with (2.4), completes the proof. □

LEMMA 2.12 (The Lindelöf property of subsets of $\mathbb{R}^n$). *Suppose $A \subseteq \mathbb{R}^n$. Let $(U_\lambda)_{\lambda\in\Lambda}$ be an indexed family of open sets in $\mathbb{R}^n$ such that $\bigcup_{\lambda\in\Lambda} U_\lambda \supseteq A$. Then there exists a* countable *subset $\Lambda_0 \subseteq \Lambda$ such that $\bigcup_{\lambda\in\Lambda_0} U_\lambda \supseteq A$.*

PROOF. The collection $\mathfrak{B}$ in Lemma 2.11 is countable. For each $a \in A$, there exists a $\gamma(a) \in \Lambda$ such that $a \in U_{\gamma(a)}$. Now apply Lemma 2.11 to $U_{\gamma(a)}$. We see that there exists some $\beta(a) \in \mathfrak{B}$ with $a \in \beta(a) \subseteq U_{\gamma(a)}$. We obtain in this way two maps $\gamma : A \to \Lambda$ and $\beta : A \to \mathfrak{B}$ such that $a \in \beta(a) \subseteq U_{\gamma(a)}$. Let $\mathfrak{B}_0$ be the image of β, which is countable since it is a subset of the countable set $\mathfrak{B}$. For each $B \in \mathfrak{B}_0$, we can choose some $\alpha(B) \in A$ with $\beta(\alpha(B)) = B$, and we obtain in this way a third map $\alpha : \mathfrak{B}_0 \to A$ such that $\beta \circ \alpha = \mathrm{Id}_{\mathfrak{B}_0}$. Now set $\Lambda_0 := (\gamma \circ \alpha)(\mathfrak{B}_0)$, which is countable since $\mathfrak{B}_0$ is countable. For each $a \in A$, we have $\gamma(\alpha(\beta(a))) \in \Lambda_0$ and

$$a \in \beta(a) = \beta(\alpha(\beta(a))) \subseteq U_{\gamma(\alpha(\beta(a)))} \subseteq \bigcup_{\lambda\in\Lambda_0} U_\lambda.$$

This completes the proof. □

We are now ready to give the proof of Theorem 2.8. Our approach is adapted from [5, p4, Theorem 1.5].

PROOF OF THEOREM 2.8. Note that $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ is an open neighborhood of M in $\mathbb{R}^d$. As will be clear in the proof, we will actually construct a partition of unity on U, instead of M, that is subordinate to $\mathcal{U} = (U_\lambda)_{\lambda\in\Lambda}$.

For each $\lambda \in \Lambda$ and $x \in U_\lambda$, there exists $\varepsilon_{\lambda,x} > 0$ with $\overline{B(x, \varepsilon_{\lambda,x})} \subseteq U_\lambda \subseteq U$. We can choose, by Lemma 2.10, a *non-negative* $f_{\lambda,x} \in C^\infty(U)$ with

$$V_{\lambda,x} := \left\{ y \in U \;\middle|\; f_{\lambda,x}(y) > 0 \right\} = B(x, \varepsilon_{\lambda,x}) \subseteq \overline{B(x, \varepsilon_{\lambda,x})} = \operatorname{supp} f_{\lambda,x} \subseteq U_\lambda.$$

Since $x \in V_{\lambda,x}$, we see that all such $V_{\lambda,x}$, as x runs through $U = \bigcup_{\lambda\in\Lambda} U_\lambda$ and λ through Λ with $x \in U_\lambda$, form an open cover of U. We may then choose among these $f_{\lambda,x}$, by Lemma 2.12, a sequence $f_1, f_2, \ldots$ of non-negative smooth functions on U such that for each n, there exists some $\lambda \in \Lambda$ with $\operatorname{supp} f_n = \overline{V_n} \subseteq U_\lambda$, and such that the sets $V_n := \{y \in U \mid f_n(y) > 0\}$, $n \geq 1$, form an open cover of U, i.e. $U = \bigcup_{n\geq 1} V_n$.

For each positive integer n, consider the set

$$W_n := \left\{ x \in U \;\middle|\; f_n(x) > 0 \text{ and } f_i(x) < \frac{1}{n} \text{ for } 1 \leq i \leq n-1 \right\},$$

and the non-negative smooth function $g_n \in C^\infty(U)$ given by

$$g_n(x) := h(f_n(x)) h\left(\frac{1}{n} - f_1(x)\right) \cdots h\left(\frac{1}{n} - f_{n-1}(x)\right),$$

where h is the function in Lemma 2.9. We have

$$\begin{aligned}\forall x \in U, \qquad g_n(x) \neq 0 &\iff h(f_n(x)) \neq 0 \text{ and } h\left(\frac{1}{n} - f_i(x)\right) \neq 0 \text{ for } 1 \leq i \leq n-1 \\ &\iff f_n(x) > 0 \text{ and } f_i(x) < \frac{1}{n} \text{ for } 1 \leq i \leq n-1 \\ &\iff x \in W_n.\end{aligned}$$

Hence W_n is open, $W_n \subseteq V_n$ and $\operatorname{supp} g_n = \overline{W_n}$.

Note that $\bigcup_{n\geq 1} V_n = U$. For each $x \in U$, there exists an $n \in \mathbb{N}^+$ such that $x \in V_n$, i.e. $f_n(x) > 0$. Set $n(x) := \min\{n \in \mathbb{N}^+ \mid f_n(x) > 0\}$. Then $x \in W_{n(x)}$, because $f_{n(x)}(x) > 0$ and $f_i(x) = 0$ for $1 \leq i \leq n(x) - 1$. Consequently,

$$\bigcup_{n\geq 1} V_n = \bigcup_{n\geq 1} W_n = U. \tag{2.5}$$

Now for every $x \in U$, consider the open set

$$W(x) := \left\{ y \in U \;\middle|\; f_{n(x)}(y) > \frac{1}{2} f_{n(x)}(x) \right\}.$$

Then $x \in W(x)$. If $W(x) \cap \overline{W_n} \neq \emptyset$ for some $n > n(x)$, then taking $y_n \in W(x) \cap \overline{W_n}$, by definition of $W(x)$ and W_n, we have

$$\frac{1}{2} f_{n(x)}(x) < f_{n(x)}(y_n) \leq \frac{1}{n}$$

so

$$n < \frac{2}{f_{n(x)}(x)}.$$

It follows that

$$\forall x \in U, \qquad \Lambda(x) := \left\{ n \in \mathbb{N}^+ \;\middle|\; W(x) \cap \overline{W_n} \neq \emptyset \right\}$$

is finite.

Define $g : U \to \mathbb{R}$ by $g(x) = \sum_{n=1}^{\infty} g_n(x)$. Note that $\operatorname{supp} g_n = \overline{W_n}$. For each $x \in U$, we have

$$y \in W(x) \implies g(y) = \sum_{n\in\Lambda(x)} g_n(y). \tag{2.6}$$

It is clear from (2.6) and the finiteness of $\Lambda(x)$ that $g \in C^\infty(U)$. Moreover, by (2.5), we have $g(x) > 0$ for every $x \in U$, and we may find, for each $n \in \mathbb{N}^+$, some $\gamma(n) \in \Lambda$ so that $\overline{W_n} \subseteq \overline{V_n} = \operatorname{supp} f_n \subseteq U_{\gamma(n)}$. This determines a map $\gamma : \mathbb{N}^+ \to \Lambda$. For each $\lambda \in \Lambda$, define

$$\begin{aligned} h_\lambda : U &\to \mathbb{R} \\ x &\mapsto \frac{1}{g(x)} \sum_{n\in\gamma^{-1}(\lambda)} g_n(x). \end{aligned} \tag{2.7}$$

We now show that $(h_\lambda)_{\lambda\in\Lambda}$ is our desired partition of unity. Since the numerator defining $h_\lambda(x)$ is a subsum of the non-negative terms defining $g(x) > 0$, we have $0 \le h_\lambda(x) \le 1$ for every $x \in U$ and $\lambda \in \Lambda$. Note first that, as in (2.6), for each $x \in U$, we have

$$\forall y \in W(x), \qquad h_\lambda(y) = \frac{1}{g(y)} \sum_{n\in\gamma^{-1}(\lambda)\cap\Lambda(x)} g_n(y). \tag{2.8}$$

By (2.8) and the finiteness of $\Lambda(x)$, we have $h_\lambda \in C^\infty(U)$.

It is clear from the definition of h_λ and $\Lambda(x)$ that

$$\begin{aligned}
\forall x \in U, \qquad W(x) \cap \operatorname{supp} h_\lambda &\subseteq \overline{\bigcup_{n\in\gamma^{-1}(\lambda)} W(x) \cap \operatorname{supp} g_n} \\
&= \overline{\bigcup_{n\in\gamma^{-1}(\lambda)} W(x) \cap \overline{W_n}} \\
&= \overline{\bigcup_{n\in\gamma^{-1}(\lambda)\cap\Lambda(x)} W(x) \cap \overline{W_n}} \\
&\subseteq \overline{\bigcup_{n\in\gamma^{-1}(\lambda)\cap\Lambda(x)} \overline{W_n}} \\
&= \bigcup_{n\in\gamma^{-1}(\lambda)\cap\Lambda(x)} \overline{W_n} \subseteq U_\lambda.
\end{aligned} \tag{2.9}$$

Since $\bigcup_{x\in U} W(x) = U$ and $\operatorname{supp} h_\lambda \subseteq U$, the above shows that

$$\forall \lambda \in \Lambda, \qquad \operatorname{supp} h_\lambda = \bigcup_{x\in U} (W(x) \cap \operatorname{supp} h_\lambda) \subseteq U_\lambda.$$

This proves the subordination condition.

For each $x \in U$, since $\Lambda(x)$ is finite, the image $\gamma(\Lambda(x))$ is a finite subset of Λ. By definition, if $W(x) \cap \operatorname{supp} h_\lambda \neq \emptyset$, then from the second line of (2.9), we deduce that there exists an $n \in \gamma^{-1}(\lambda)$ with $W(x) \cap \overline{W_n} \neq \emptyset$, so that $n \in \Lambda(x)$ and $\lambda = \gamma(n)$ is in the finite subset $\gamma(\Lambda(x)) \subseteq \Lambda$. This establishes the local finiteness condition.

Finally, for each $x \in U$, in particular for each $x \in M \subseteq U$, we have

$$\begin{aligned}
\sum_{\lambda\in\Lambda} h_\lambda(x) &= \sum_{\lambda\in\Lambda} h_\lambda|_{W(x)}(x) \\
&= \frac{1}{g(x)} \sum_{\lambda\in\Lambda} \left(\sum_{n\in\gamma^{-1}(\lambda)\cap\Lambda(x)} g_n(x) \right) \\
&= \frac{1}{g(x)} \sum_{n\in\Lambda(x)} g_n(x) \\
&= \frac{1}{g(x)} g(x) = 1.
\end{aligned}$$

This proves the unity condition. The proof of Theorem 2.8 is now complete. □

Exercises

2.1. Prove Proposition 2.5.

2.2. Consider $\mathbb{R}^\infty = \{(x_n)_{n\geq 1}\}$, the real vector space of all sequences of real numbers. For each $n \in \mathbb{N}^+$, we identify $\mathbb{R}^n$ with the linear subspace of $\mathbb{R}^\infty$ consisting of all sequences whose terms starting from the $n+1$-th are all zero, i.e.

$$\mathbb{R}^n := \left\{(x_k)_{k\geq 1} \;\middle|\; \forall k \geq n+1, \quad x_k = 0\right\}.$$

Suppose $m, n \in \mathbb{N}^+$ with $m \leq n$. We then have $\mathbb{R}^m \subseteq \mathbb{R}^n$ according to the above identification. Let $A \subseteq \mathbb{R}^m \subseteq \mathbb{R}^n$, $k \in \mathbb{N}_\infty$, and suppose $f : A \to \mathbb{K}$ is a function.

(1) Show that for any $a \in A$, the following are equivalent:
  (1.a) There exists an open neighborhood U_a of a in $\mathbb{R}^m$ and a $g_a \in C^k(U_a)$ such that $f|_{U_a \cap A} = g_a|_{U_a \cap A}$.
  (1.b) There exists an open neighborhood V_a of a in $\mathbb{R}^n$ and an $h_a \in C^k(V_a)$ such that $f|_{V_a \cap A} = h_a|_{V_a \cap A}$.

  Remark: this means the notion of f being of class C^k at $a \in A$ is independent of the choice of d as long as $A \subseteq \mathbb{R}^d$.

(2) Show that the following are equivalent:
  (2.a) There exists an open neighborhood U of A in $\mathbb{R}^m$, and a $g \in C^k(U)$ such that $g|_A = f$.
  (2.b) There exists an open neighborhood V of A in $\mathbb{R}^n$, and a $h \in C^k(V)$ such that $h|_A = f$.

  Remark: this means the notion of f being of class C^k on A is independent of the choice of d as long as $A \subseteq \mathbb{R}^d$.

2.3. (**Local finiteness**) Let (X, d) be a metric space, and $(A_\lambda)_{\lambda\in\Lambda}$ an indexed family of subsets of X. We say $(A_\lambda)_{\lambda\in\Lambda}$ is **locally finite** on X if for each $x \in X$, there exists an open neighborhood U_x of x in X such that

$$\{\lambda \in \Lambda \mid U_x \cap A_\lambda \neq \emptyset\}$$

is finite.

(1) Show that if $(A_\lambda)_{\lambda\in\Lambda}$ is locally finite on X, then taking closures commutes with taking unions of $(A_\lambda)_{\lambda\in\Lambda}$, i.e.

$$\overline{\bigcup_{\lambda\in\Lambda} A_\lambda} = \bigcup_{\lambda\in\Lambda} \overline{A_\lambda}.$$

(2) Dropping the local finiteness assumption, give a counterexample that demonstrates the failure of commutation of taking closures and taking unions.

  Hint: one possible example goes as follows: take any non-empty subset $A \subseteq \mathbb{R}$ that is not closed, and consider the indexed family $(\{a\})_{a\in A}$.

(3) Show that if $(f_\lambda)_{\lambda\in\Lambda}$ is a family of continuous functions from X to $\mathbb{K}$ such that $(\operatorname{supp} f_\lambda)_{\lambda\in\Lambda}$ is locally finite, then

$$f := \sum_{\lambda\in\Lambda} f_\lambda$$

is a well-defined *continuous* function from X to $\mathbb{K}$.

2.4. Let $n \in \mathbb{N}^+$ and $A \subseteq \mathbb{R}^n$. The goal of this exercise is to show that for any continuous function $f : A \to \mathbb{K}$ and any $\varepsilon > 0$, there exists a smooth function g on A such that $|f(a) - g(a)| < \varepsilon$ for any $a \in A$.

(1) Show that for every $a \in A$, there exists an open neighborhood U_a of a in $\mathbb{R}^n$ such that for any $x \in U_a \cap A$, we have $|f(x) - f(a)| < \varepsilon$.
(2) Consider the open cover $\mathcal{U} := (U_a)_{a\in A}$ of A and let $(\varphi_a)_{a\in A}$ be a partition of unity on A subordinate to $\mathcal{U}$. Show that

$$g := \sum_{a\in A} f(a)\varphi_a$$

is a well-defined smooth function on A.
(3) Show that $|f(a) - g(a)| < \varepsilon$ for every $a \in A$.

2.5. This exercise, while not particularly difficult, may involve some additional reading. Let $I = [0, 1]$ be the unit interval, and $I^2 \subseteq \mathbb{R}^2$ the unit cube.

(1) Show that there exists a continuous map $f : I \to I^2$ that is surjective.
Hint: you may search for “the Peano curve” if you get stuck.
(2) Show, using partitions of unity, that for any continuous map $f : I \to I^2$ and any $\varepsilon > 0$, there exists a smooth map $g : I \to \mathbb{R}^2$ with $g(I) \subseteq I^2$ such that $|g(t) - f(t)| < \varepsilon$ for every $t \in I$, and $g(I) \subsetneq I^2$.
Remark: there is a general result called Sard’s lemma, which implies that there is no surjective smooth map from I onto I^2.

3. Manifolds: charts, zeros and graphs

Fix $k \in \mathbb{N}^+_\infty$. We treat manifolds of class C^k uniformly; **smooth** means $k = \infty$. For subsets of Euclidean spaces, a map is of class C^k if each component has a C^k extension to an open neighborhood of its domain, as in Definition 2.2. The first two exercises in Section 1 establish the algebra and composition rules for C^k functions on open sets. These rules also hold on arbitrary subsets: sums and products are obtained by taking sums and products of extensions. For the composition, let $A \subseteq \mathbb{R}^l$ and $B \subseteq \mathbb{R}^m$, and let $f : A \to B$ and $g : B \to \mathbb{R}^n$ be of class C^k. Choose open neighborhoods $U \subseteq \mathbb{R}^l$ of A and $V \subseteq \mathbb{R}^m$ of B, and C^k extensions $\widetilde{f} : U \to \mathbb{R}^m$, $\widetilde{g} : V \to \mathbb{R}^n$ with $\widetilde{f}\big|_A = f$ and $\widetilde{g}|_B = g$. The set $W = \widetilde{f}^{-1}(V) \subseteq U$ is open in $\mathbb{R}^l$ and contains A, since $f(A) \subseteq B \subseteq V$. Hence

$$\widetilde{g} \circ \widetilde{f}\big|_W : W \to \mathbb{R}^n$$

is of class C^k, and its restriction to A equals $g \circ f$. Maps of class C^k on subsets are continuous, since their extensions are continuous. We allow $\mathbb{R}^0 = \{0\}$.

DEFINITION 3.1. Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$. A C^k**-diffeomorphism** from A to B is a bijection $f : A \to B$ such that both f and f^{-1} are of class C^k. The sets are C^k-**diffeomorphic** if such a map exists. Every C^k-diffeomorphism is a homeomorphism. An unqualified **diffeomorphism** means a smooth one.

EXAMPLE 3.2. The closed triangle and closed disk

$$\Delta_2 = \left\{(x_0, x_1, x_2) \in \mathbb{R}^3_{\geq 0} \;\middle|\; x_0 + x_1 + x_2 = 1\right\}, \qquad B_2 = \left\{x \in \mathbb{R}^2 \;\middle|\; |x| \leq 1\right\}$$

are homeomorphic but not C^1-diffeomorphic, hence not C^k-diffeomorphic for any $k \geq 1$. Their interiors are smoothly diffeomorphic, where the interior of Δ_2 is taken in its affine plane. Exercise 3.4 proves these assertions by elementary methods; the obstruction uses only first derivatives.

DEFINITION 3.3. A C^k**-manifold of dimension** $d \in \mathbb{N}$ is a nonempty subset $M \subseteq \mathbb{R}^n$ such that every $p \in M$ has a relatively open neighborhood $U \subseteq M$ and a C^k-diffeomorphism $\varphi : U \to V$, where $V \subseteq \mathbb{R}^d$ is open. The pair (U, φ) is a C^k **chart**, and $\varphi^{-1} : V \to U$ is a **local parametrization**. The coordinates $\varphi(q)$ describe points $q \in U$, not necessarily all of M; different neighborhoods may require different charts. For $k = \infty$, we call M a **smooth manifold**. We call $n - d$ the **codimension** in $\mathbb{R}^n$.

DEFINITION 3.4. A **topological manifold of dimension** $d \in \mathbb{N}$ is a nonempty second-countable Hausdorff space M such that every $p \in M$ has an open neighborhood $U \subseteq M$ and a homeomorphism $\varphi : U \to V$, where $V \subseteq \mathbb{R}^d$ is open. It can be regarded as a C^0**-manifold**: the charts and their inverses are continuous, with no differentiability required. Every C^k-manifold defined above is a topological manifold with its subspace topology.

REMARK 3.5 (Uniqueness of dimension). The dimension of a nonempty C^k-manifold is well-defined. Indeed, suppose two charts $\varphi : U \to U'$ and $\psi : V \to V'$ at the same point p have images open in $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively. Their overlap $U \cap V$ is a relatively open neighborhood of p, so $\varphi(U \cap V)$ and $\psi(U \cap V)$ are nonempty open subsets of the respective Euclidean spaces. The map

$$F = \psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

and its inverse $H = \varphi \circ \psi^{-1}$ are of class C^k, hence C^1. Set $a = \varphi(p)$ and $b = \psi(p)$. Differentiating the two inverse identities gives

$$\mathrm{d}H(b)\,\mathrm{d}F(a) = I_d, \qquad \mathrm{d}F(a)\,\mathrm{d}H(b) = I_e.$$

The first identity makes $\mathrm{d}F(a) : \mathbb{R}^d \to \mathbb{R}^e$ injective, so $d \leq e$; the second makes $\mathrm{d}H(b)$ injective, so $e \leq d$. Thus $d = e$.

For topological manifolds, whose charts are merely homeomorphisms, uniqueness of dimension is significantly more difficult. It requires nontrivial topological results, such as Brouwer's invariance of domain theorem. One consequence is that nonempty open subsets of $\mathbb{R}^d$ and $\mathbb{R}^e$ can be homeomorphic only if $d = e$, which gives the corresponding conclusion for topological charts.

Remark 3.6. Let $M \subseteq \mathbb{R}^m$ be a C^k-manifold of dimension d, and let $n \geq m$. Under the standard inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$, $x \mapsto (x, 0)$, M is also a C^k-manifold of dimension d in $\mathbb{R}^n$. Indeed, the subspace topology on M is unchanged, and Exercise 2.2, applied to each component of a chart, shows that the chart remains of class C^k. Its inverse remains of class C^k after adjoining zero components. Thus the same charts verify the definition in $\mathbb{R}^n$.

Remark 3.7. See Exercise 3.1 for a characterization of manifolds of dimension zero. If (U, φ) and (V, ψ) are C^k charts of a d-dimensional manifold, then $\varphi(U \cap V)$ and $\psi(U \cap V)$ are open in $\mathbb{R}^d$. The **transition map**

$$\psi|_{U\cap V} \circ \varphi^{-1}\big|_{\varphi(U\cap V)} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a C^k-diffeomorphism. Its inverse is obtained by interchanging φ and ψ. It changes local coordinates only on the overlap $U \cap V$, where both charts are defined.

In dimensions at most three, every topological manifold is homeomorphic to a smooth manifold; the three-dimensional case is a deep theorem; see [3, Section 8.3]. In dimensions at least four, smoothability can fail. A special theorem in dimension four says that a connected topological four-manifold becomes smoothable after removal of one point; see [6, Corollary 2.2.3] and [3, Section 8.2]. The corresponding assertion in all higher dimensions is false. These facts are included only for cultural background and are not required in this course.

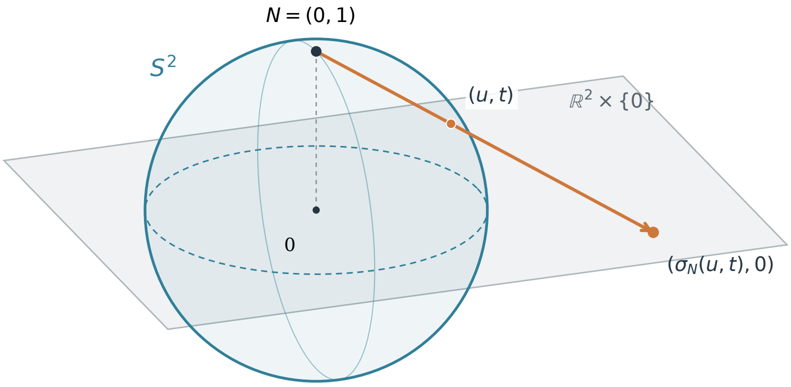
Example 3.8 (Spheres). Let $d \geq 1$ and $S^d = \{(u, t) \in \mathbb{R}^d \times \mathbb{R} \mid |u|^2 + t^2 = 1\}$. Set $N = (0, 1) \in \mathbb{R}^d \times \mathbb{R}$, where 0 denotes the zero vector of $\mathbb{R}^d$, and define

$$\sigma_N : S^d \setminus \{N\} \to \mathbb{R}^d, \qquad \sigma_N(u, t) = \frac{u}{1 - t}.$$

Its inverse is

$$v \longmapsto \left(\frac{2v}{1 + |v|^2}, \frac{|v|^2 - 1}{1 + |v|^2} \right).$$

Substitution verifies both inverse identities. Both maps are smooth; the projection extends to $t \neq 1$. Projection from $(0, -1) \in \mathbb{R}^d \times \mathbb{R}$, given by $(u, t) \mapsto u/(1 + t)$, supplies the other chart. These two maps are called **stereographic projections**. Each chart omits its projection pole, but their domains together cover S^d. Thus they provide local coordinates near every point, without a single chart covering the whole sphere. Thus S^d is a smooth manifold of dimension d, hence a C^k-manifold for every $k \in \mathbb{N}^+_\infty$. Also $S^0 = \{-1, 1\}$ is a zero-dimensional smooth manifold.



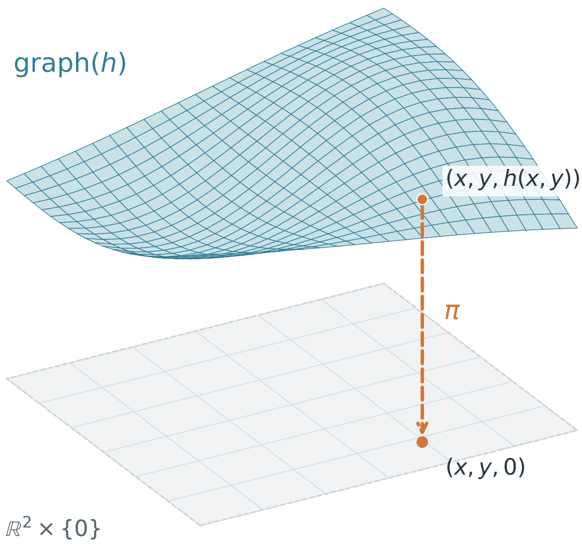
Stereographic projection of S^2: identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$. The line through N and (u, t) meets this plane at $(\sigma_N(u, t), 0)$.

Example 3.9 (Graphs). Let $U \subseteq \mathbb{R}^d$ be nonempty and open, and let $h : U \longrightarrow \mathbb{R}^m$ be of class C^k. Then

$$\operatorname{graph}(h) = \left\{(u, h(u)) \in \mathbb{R}^d \times \mathbb{R}^m \;\middle|\; u \in U\right\}$$

is a C^k-manifold of dimension d. Indeed, the map $U \longrightarrow \operatorname{graph}(h)$, $u \mapsto (u, h(u))$, is a C^k bijection. Its inverse is the restriction to $\operatorname{graph}(h)$ of the linear projection

$$\pi : \mathbb{R}^d \times \mathbb{R}^m \longrightarrow \mathbb{R}^d, \qquad \pi(u, v) = u.$$



A portion of a smooth graph in $\mathbb{R}^3$. The chart π sends $(x, y, h(x, y))$ to (x, y), identifying $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}$. Its inverse lifts (x, y) to the surface.

For $m = 0$, this says that every nonempty open subset of $\mathbb{R}^d$ is a C^k-manifold of dimension d.

We recall the two local inversion theorems from last semester. Their C^1 versions are assumed; the higher regularity follows below. Their conclusions hold on suitable neighborhoods of the specified point: neither a global inverse on the original domain nor a graph description of the entire zero set is asserted.

THEOREM 3.10 (Inverse function theorem). *Let $k \in \mathbb{N}_\infty^+$, let $\Omega \subseteq \mathbb{R}^n$ be open, and let $f \in C^k(\Omega, \mathbb{R}^n)$. If $a \in \Omega$ and $\det \mathrm{d}f(a) \neq 0$, there are open neighborhoods U of a and V of $f(a)$ such that $f : U \to V$ is bijective and its inverse g is of class C^k. Moreover,*

$$\mathrm{d}g(y) = \big(\mathrm{d}f(g(y))\big)^{-1}, \qquad y \in V.$$

PROOF. The C^1 theorem gives U, V, g; differentiating $f \circ g = \mathrm{Id}_V$ gives the displayed formula. Matrix inversion is smooth on the set of invertible matrices, since $A^{-1} = (\det A)^{-1} \operatorname{adj} A$, and determinants and cofactors are polynomials in the entries. If f is C^k and g is C^j with $1 \le j < k$, the formula makes $\mathrm{d}g$ of class C^j, so g is C^{j+1}. Induction proves the finite cases and, by taking every finite order, the smooth case. □

THEOREM 3.11 (Implicit function theorem). *Let $k \in \mathbb{N}_\infty^+$, let $\Omega \subseteq \mathbb{R}^d \times \mathbb{R}^m$ be open, and let $f \in C^k(\Omega, \mathbb{R}^m)$. Suppose $f(a, b) = 0$ and the partial derivative $\mathrm{d}_v f(a, b)$ is invertible. There are open neighborhoods A of a and B of b, with $A \times B \subseteq \Omega$, and a unique $h \in C^k(A, B)$ such that*

$$f(u, v) = 0 \quad \Longleftrightarrow \quad v = h(u) \qquad ((u, v) \in A \times B).$$

In particular, $h(a) = b$, and

$$\mathrm{d}h(u) = -\big(\mathrm{d}_v f(u, h(u))\big)^{-1} \mathrm{d}_u f(u, h(u)).$$

PROOF. The C^1 theorem gives A, B, h. Shrink the neighborhoods so that $\mathrm{d}_v f$ is invertible throughout their product. Differentiating $f(u, h(u)) = 0$ gives the formula. The same induction using smooth matrix inversion as in the preceding proof gives $h \in C^k$. □

DEFINITION 3.12. A C^k map $f : \Omega \to \mathbb{R}^m$, with $\Omega \subseteq \mathbb{R}^n$ open, is a **submersion at** $p \in \Omega$ if $\mathrm{d}f(p) : \mathbb{R}^n \to \mathbb{R}^m$ is surjective, or equivalently has rank m. It is a **submersion** if this holds at every point. Necessarily $m \le n$ when $\Omega \neq \emptyset$.

EXAMPLE 3.13 (Regular level sets). If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a C^k submersion and $f^{-1}(0) \neq \emptyset$, then its zero set is a C^k-manifold of dimension $n - m$, hence codimension m. Indeed, at a zero p, choose m columns of $\mathrm{d}f(p)$ giving an invertible minor. After permuting coordinates, write $p = (a, b) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$. The implicit function theorem supplies open neighborhoods A of a and B of b on whose product the zero set is a C^k graph. This describes the level set locally near p; another point may require a different coordinate permutation and a different graph.

More generally, $f^{-1}(c)$ is a **regular level set** if $\mathrm{d}f(p)$ is surjective for every $p \in f^{-1}(c)$. Applying the same argument to $f - c$ shows that every nonempty regular level set is a C^k-manifold of dimension $n - m$. The submersion condition ensures the existence of these local graphs; without it, a level set need not be a C^k-manifold. Exercise 3.3 gives counterexamples and shows that a singular defining equation can also have a smooth zero set.

The next theorem characterizes manifolds by several local descriptions. The neighborhoods, coordinate permutations, and maps in its conditions may depend on p. In particular, it does not assert that the whole manifold is one graph or the zero set of one globally defined submersion.

THEOREM 3.14 (Local descriptions of a C^k-manifold). *Let $k \in \mathbb{N}^+_\infty$, $\emptyset \neq M \subseteq \mathbb{R}^n$, and $0 \leq d \leq n$. The following are equivalent:*

(i) *M is a C^k-manifold of dimension d.*

(ii) *At every $p \in M$, after a permutation of coordinates, there are open sets $U \subseteq \mathbb{R}^d$, $V \subseteq \mathbb{R}^{n-d}$, and a map $h \in C^k(U, V)$ such that $p \in U \times V$ and*
$$M \cap (U \times V) = \operatorname{graph}(h).$$

(iii) *Every $p \in M$ has an open neighborhood $W \subseteq \mathbb{R}^n$ and a C^k-diffeomorphism $\Phi : W \longrightarrow W'$, with $W' \subseteq \mathbb{R}^n$ open, such that*
$$\Phi(M \cap W) = W' \cap (\mathbb{R}^d \times \{0\}).$$
*Such a map is called a **straightening chart**.*

(iv) *Every $p \in M$ has an open neighborhood $W \subseteq \mathbb{R}^n$ and a C^k submersion $F : W \longrightarrow \mathbb{R}^{n-d}$ such that $M \cap W = F^{-1}(0)$.*

PROOF. We prove the implications in the order stated, then return to the first condition. Fix $p \in M$.

(i) implies (ii). Choose a chart $\varphi : \Omega \longrightarrow U'$, where Ω is relatively open in M, $p \in \Omega$, and $U' \subseteq \mathbb{R}^d$ is open. Set $f = \varphi^{-1}$ and $a = \varphi(p)$. By the extension definition of class C^k, φ extends to a map $G : \widetilde{\Omega} \longrightarrow \mathbb{R}^d$ of class C^k on an open neighborhood $\widetilde{\Omega}$ of Ω in $\mathbb{R}^n$. Since $f(U') = \Omega$, $G \circ f = \mathrm{Id}_{U'}$. The chain rule gives
$$\mathrm{d}G(p)\, \mathrm{d}f(a) = I_d.$$
Thus $\mathrm{d}f(a) : \mathbb{R}^d \longrightarrow \mathbb{R}^n$ is injective: if $\mathrm{d}f(a)z = 0$, multiplying by $\mathrm{d}G(p)$ gives $z = 0$. Consequently its matrix has d independent rows. Permute coordinates so these are the first d rows, and let π_1 and π_2 denote projection onto the first d and last $n - d$ coordinates, respectively. Then $\mathrm{d}(\pi_1 \circ f)(a)$ is invertible.

By the C^k inverse function theorem, there are open neighborhoods $U_0' \subseteq U'$ of a and $U_0 \subseteq \mathbb{R}^d$ of $\pi_1(p)$ such that $q = (\pi_1 \circ f)|_{U_0'} : U_0' \longrightarrow U_0$ is a C^k-diffeomorphism. Define
$$h = \pi_2 \circ f \circ q^{-1} : U_0 \longrightarrow \mathbb{R}^{n-d}.$$

This map is of class C^k, and $f \circ q^{-1}(u) = (u, h(u))$, so $f(U_0') = \mathrm{graph}(h)$. We must still exclude other points of M near this graph. Because f is a homeomorphism and Ω is relatively open in M, $f(U_0')$ is relatively open in M. Hence $f(U_0') = M \cap W$ for some open $W \subseteq \mathbb{R}^n$.

Write $p = (u_0, v_0)$, so $h(u_0) = v_0$. Choose open neighborhoods $U_1 \subseteq U_0$ of u_0 and $V \subseteq \mathbb{R}^{n-d}$ of v_0 such that $U_1 \times V \subseteq W$. By continuity of h, shrink U_1 to an open neighborhood U with $h(U) \subseteq V$. If $(u,v) \in M \cap (U \times V)$, then it belongs to $M \cap W = \mathrm{graph}(h)$, so $v = h(u)$. Conversely, for every $u \in U$, the point $(u, h(u))$ belongs to $f(U_0') \subseteq M$ and to $U \times V$. This proves the required equality, with h restricted to U.

(ii) implies (iii). In the coordinates of the graph description, set $W = U \times V$ and define $\Phi(u,v) = (u, v - h(u))$. Its image is

$$W' = \left\{ (u,w) \in U \times \mathbb{R}^{n-d} \;\middle|\; w + h(u) \in V \right\}.$$

This set is open, since $(u,w) \mapsto w + h(u)$ is continuous and V is open. The map $\Psi(u,w) = (u, w + h(u))$ sends W' to W, and direct substitution gives $\Psi \circ \Phi = \mathrm{Id}_W$ and $\Phi \circ \Psi = \mathrm{Id}_{W'}$. Both maps are of class C^k. Moreover, the last coordinates of $\Phi(u,v)$ vanish exactly when $v = h(u)$. Thus the asserted straightening equality holds. Composing with the coordinate permutation, which is a linear diffeomorphism, gives a straightening map in the original coordinates.

(iii) implies (iv). Let $\pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-d}$ be projection onto the last coordinates and set $F = \pi_2 \circ \Phi$. Then $F \in C^k(W, \mathbb{R}^{n-d})$, and for $x \in W$,

$$F(x) = 0 \quad \Longleftrightarrow \quad \Phi(x) \in W' \cap (\mathbb{R}^d \times \{0\}) \quad \Longleftrightarrow \quad x \in M \cap W.$$

Differentiating the inverse identities for Φ shows that $\mathrm{d}\Phi(x)$ is invertible at every $x \in W$. Hence $\mathrm{d}F(x) = \pi_2 \circ \mathrm{d}\Phi(x)$ is surjective: for any $z \in \mathbb{R}^{n-d}$, it sends $(\mathrm{d}\Phi(x))^{-1}(0,z)$ to z. Thus F is a submersion throughout W, not only on its zero set.

(iv) implies (i). The matrix $\mathrm{d}F(p)$ has $n-d$ independent columns. Permute coordinates so these are the last columns and write $p = (a,b) \in \mathbb{R}^d \times \mathbb{R}^{n-d}$. Then $F(a,b) = 0$ and $\mathrm{d}_v F(a,b)$ is invertible. The C^k implicit function theorem supplies open neighborhoods U of a and V of b, with $U \times V \subseteq W$, and $h \in C^k(U,V)$ such that

$$M \cap (U \times V) = \{(u,v) \in U \times V \mid F(u,v) = 0\} = \mathrm{graph}(h).$$

Set $\Omega = M \cap (U \times V)$, a relatively open neighborhood of p in M. The projection $\varphi : \Omega \to U$, $\varphi(u, h(u)) = u$, is bijective with inverse $u \mapsto (u, h(u))$. The latter is C^k on the open set U, and φ is the restriction of a linear projection on $\mathbb{R}^n$. Thus both maps are C^k in our extension sense, and (Ω, φ) is a chart. The coordinate permutation can again be included in this chart without changing its regularity.

These arguments also cover $d = 0$ and $d = n$, with the convention $\mathbb{R}^0 = \{0\}$. For $d = 0$, a chart image and a local graph are single points, and the inverse function theorem isolates each zero in the last implication. For $d = n$, the graph has no last coordinates and the zero set of a map to $\mathbb{R}^0$ is its whole domain; the local model is an open subset of $\mathbb{R}^n$. □

REMARK 3.15. In the language of abstract manifolds, the straightening condition (iii) says that M is an **embedded** C^k **submanifold** of $\mathbb{R}^n$. It concerns how M sits inside $\mathbb{R}^n$: locally, after a change of coordinates on an open subset of $\mathbb{R}^n$, the inclusion of M becomes $u \mapsto (u, 0)$. In particular, the topology on M is the subspace topology, and the coordinates straighten the entire nearby part of M.

An abstract C^k-manifold instead starts with a second-countable Hausdorff space and charts into $\mathbb{R}^d$ whose transition maps are C^k-diffeomorphisms; no containing Euclidean space is specified. Our charts give exactly this structure on M, while the theorem also establishes its compatibility with the given inclusion into $\mathbb{R}^n$. Thus the chart description already leads toward the **intrinsic point of view**, in which one works with local coordinates on the manifold itself. Abstract smooth manifolds are not required in this course; this remark only explains the relation between the two descriptions.

REMARK 3.16. By convention, the empty set is a manifold of dimension -1. This agrees with the convention in topological dimension theory; see [4, Definition III.1, p. 24]. The definition and theorem above concern nonempty manifolds.

For a nonempty manifold in $\mathbb{R}^n$, its dimension d does not exceed n: the identity $\mathrm{d}G(p)\,\mathrm{d}f(a) = I_d$ in the proof above makes $\mathrm{d}f(a) : \mathbb{R}^d \to \mathbb{R}^n$ injective.

For us, C^k-manifolds with finite k are a useful technical variation, but our main focus remains the smooth case. In many proofs we then need not keep track of the available regularity, and the ideas of the proofs become more transparent. Unless stated otherwise, "manifold" and "diffeomorphism" will therefore mean smooth from now on; in particular, the following exercises use the smooth convention.

Exercises

3.1. Show that a nonempty subset $M \subseteq \mathbb{R}^n$ is a zero-dimensional manifold if and only if it is discrete in its subspace topology, that is, every $p \in M$ has an open neighborhood $W \subseteq \mathbb{R}^n$ with $W \cap M = \{p\}$.

3.2.
(1) Fix $R > r > 0$. Show that

$$T = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; (x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2(x^2 + y^2)\right\}$$

is a compact smooth manifold of dimension two. Hint: first show that $\rho = \sqrt{x^2 + y^2} > 0$ on T, and rewrite the equation as $(\rho - R)^2 + z^2 = r^2$. Check that the gradient of this defining function does not vanish on T.

(2) Fix $R > a > 0$, and define

$$p(\theta, t) = \big((R + t\cos(\theta/2))\cos\theta, (R + t\cos(\theta/2))\sin\theta, t\sin(\theta/2)\big).$$

Let $M = p(\mathbb{R}\times(-a, a))$, the **open Möbius band**. Verify $p(\theta+2\pi, t) = p(\theta, -t)$. For each θ_0, choose an open interval J containing θ_0 of length less than 2π.

Show that the restriction of p to $J \times (-a, a)$ is a diffeomorphism onto a relatively open subset of M. Hint: use a smooth branch $\theta \in J$ of the polar angle and the formula $t = (\rho - R)\cos(\theta/2) + z\sin(\theta/2)$. Deduce that M is a two-dimensional manifold.

The torus with $R = 2$, $r = 0.7$, and the open Möbius band with $R = 2$, $a = 0.65$. The edge of the pictured band is not included.

3.3. (1) Show that the zero set of $f(x, y) = xy$ is not a manifold near the origin. Hint: away from the origin its dimension would have to be one; deleting the origin from a small connected chart would leave two components, whereas the crossing has four branches.

(2) Show that $C = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; y^2 = x^3\right\}$ is homeomorphic to $\mathbb{R}$ via $t \mapsto (t^2, t^3)$, but is not a smooth manifold at the origin. Hint: for any smooth curve $\gamma(s) = (x(s), y(s))$ in C with $\gamma(0) = 0$, use $x(s) \geq 0$ and $|y(s)| = x(s)^{3/2}$ to show $\gamma'(0) = 0$. Compare with the derivative of a local parametrization.

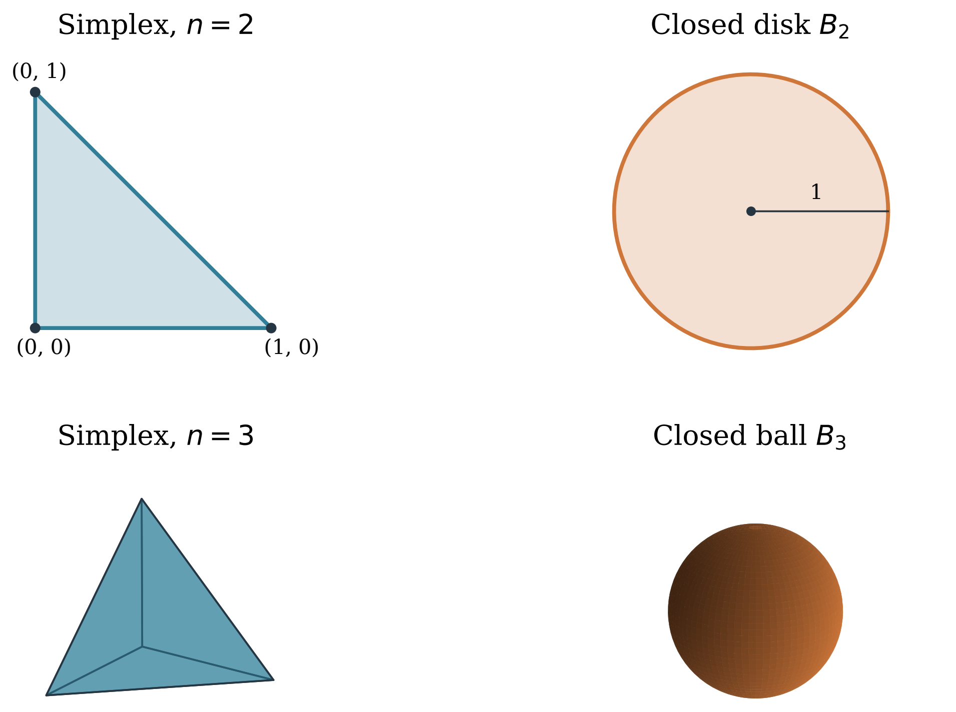
(3) Show that $f(x, y) = y^2$ has derivative zero at every point of its zero set, although that set is a smooth manifold. Explain why regularity of a particular defining equation is sufficient but not necessary.

3.4. For $n \geq 2$, let

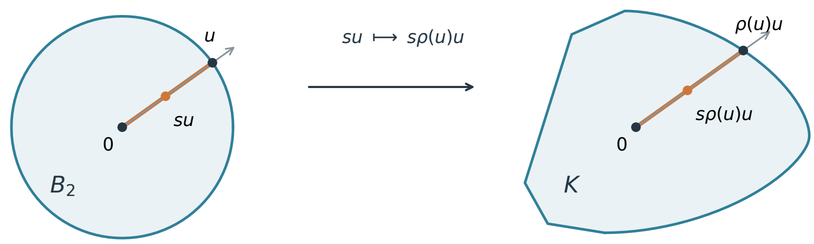
$$\Delta_n = \left\{x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \sum_{i=0}^{n} x_i = 1\right\}, \qquad B_n = \{y \in \mathbb{R}^n \mid |y| \leq 1\}.$$

Interiors and boundaries of Δ_n are taken relative to $H = \left\{x \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1\right\}$.

(1) Show that every compact convex set $K \subseteq \mathbb{R}^n$ with nonempty interior is homeomorphic to B_n. Hint: translate so that $0 \in \operatorname{Int} K$, choose $0 < r < R$ with $rB_n \subseteq K \subseteq RB_n$, and define $p(x) = \inf\{t > 0 \mid x \in tK\}$. Prove subadditivity and $|p(x) - p(y)| \leq |x - y|/r$. The radial function $\rho(u) = 1/p(u)$ on S^{n-1} is continuous; use $su \mapsto s\rho(u)u$, $0 \leq s \leq 1$.



The simplex and closed unit ball in dimensions two and three. The simplices are shown in the coordinates of the affine map a from part (2).



On each ray, the map in (1) sends the segment from 0 to u onto the segment from 0 to $\rho(u)u$, preserving the fraction s of its length. Here the boundary of K consists of straight segments and a curved arc of varying curvature.

(2) The affine map $a(y_1, \ldots, y_n) = (1 - \sum_{i=1}^n y_i, y_1, \ldots, y_n)$ maps $\mathbb{R}^n$ bijectively onto H. Show that it restricts to a diffeomorphism from $K = \left\{y \in \mathbb{R}^n_{\geq 0} \;\middle|\; \sum_{i=1}^n y_i \leq 1\right\}$ to Δ_n, and deduce a homeomorphism $\Delta_n \cong B_n$.

(3) Suppose that $\varphi : \Delta_n \longrightarrow B_n$ is a diffeomorphism. Show that it maps interiors onto interiors, and hence restricts to a diffeomorphism of boundaries. Hint: use a to work in $\mathbb{R}^n$. At an interior point, differentiate the inverse identities for

smooth extensions and apply the inverse function theorem. Apply the same argument to the inverse map.

(4) For a vertex e_i of Δ_n, prove that every smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \partial\Delta_n$ with $\gamma(0) = e_i$ satisfies $\gamma'(0) = 0$. Hint: each coordinate other than the i-th has a local minimum, and the sum of the coordinates is constant. Every point of $\partial B_n = S^{n-1}$ lies on a smooth curve with nonzero velocity. Use the chain rule to rule out a diffeomorphism of the boundaries, and hence of the closed sets. Explain why $n = 1$ is an exception.

(5) Show that the interiors are nevertheless diffeomorphic. Verify that

$$(t_1, \ldots, t_n) \longmapsto \frac{(1, e^{t_1}, \ldots, e^{t_n})}{1 + \sum_{i=1}^n e^{t_i}}$$

is a diffeomorphism from $\mathbb{R}^n$ onto $\operatorname{Int} \Delta_n$, with inverse

$$x \longmapsto (\log(x_1/x_0), \ldots, \log(x_n/x_0)).$$

Combine it with the diffeomorphism $t \mapsto t/\sqrt{1 + |t|^2}$ from $\mathbb{R}^n$ to $\operatorname{Int} B_n$, whose inverse is $y \mapsto y/\sqrt{1 - |y|^2}$.

3.5. Let $M \subseteq \mathbb{R}^n$ be a nonempty *connected* one-dimensional manifold. A smooth curve is called of **unit speed** if $|\gamma'(t)| = 1$.

(1) For a local parametrization $f : (a, b) \longrightarrow M$, show that $f'(t) \neq 0$. Reparametrize by $s(t) = \int_{t_0}^{t} |f'(u)| \, \mathrm{d}u$ to obtain a unit speed parametrization. Show that the coordinate changes between such parametrizations have the form $s \mapsto s + c$ or $s \mapsto -s + c$ on each connected overlap.

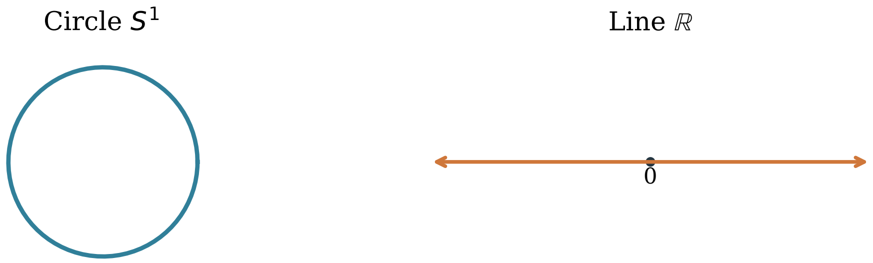
(2) Fix a point and one of its two unit velocities. By joining compatible local parametrizations, construct a maximal unit speed curve $\gamma : I \longrightarrow M$, where I is an open interval containing zero. Prove uniqueness for these initial data. Hint: in a unit speed coordinate a curve has coordinate derivative constantly 1 or constantly -1.

(3) Show that $\gamma(I) = M$. Hint: the image is open. If p is in its closure, choose a unit speed chart $f : (-2\varepsilon, 2\varepsilon) \longrightarrow M$ centered at p, and a point of the image in $f((-\varepsilon, \varepsilon))$. Continue through that chart in both directions; maximality forces the curve to reach p. Thus the image is also closed.

(4) If $\gamma(s) = \gamma(t)$ for $s < t$, show that their velocities agree. Hint: opposite velocities and uniqueness would give $\gamma(s + u) = \gamma(t - u)$ up to the midpoint, contradicting unit speed there. Deduce that $I = \mathbb{R}$ and that γ is periodic. Show that its periods form $L\mathbb{Z}$ for some $L > 0$, using local injectivity and closedness of the period group. Prove that $(\cos(2\pi t/L), \sin(2\pi t/L)) \mapsto \gamma(t)$ is a diffeomorphism from S^1 onto M.

(5) If γ is injective, show that it is a diffeomorphism $I \longrightarrow M$, and recall that every nonempty open interval is diffeomorphic to $\mathbb{R}$. Conclude that M is diffeomorphic to S^1 when compact, and to $\mathbb{R}$ when noncompact. Finally, show that an

arbitrary one-dimensional manifold has at most countably many connected components, each open and of one of these two types.



The two types of nonempty connected one-dimensional manifolds.

3.6. Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be smooth, with Ω open. Suppose $a \in \Omega$ has an open neighborhood $\Omega_0 \subseteq \Omega$ such that $\operatorname{rank} \mathrm{d}f(x) = r$ for every $x \in \Omega_0$. Prove that there are open neighborhoods $U \subseteq \Omega_0$ of a and $V \subseteq \mathbb{R}^m$ of $f(a)$, with $f(U) \subseteq V$, and diffeomorphisms $\alpha : U \to U'$, $\beta : V \to V'$ onto open sets such that $\alpha(a) = 0$, $\beta(f(a)) = 0$, and

$$(\beta \circ f \circ \alpha^{-1})(u, v) = (u, 0) \quad \text{for every } (u, v) \in U' \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r}.$$

(1) Translate both points to zero. If $r > 0$, permute source and target coordinates so that the matrix

$$\frac{\partial(f_1, \ldots, f_r)}{\partial(x_1, \ldots, x_r)}$$

is invertible at zero. Apply the inverse function theorem to

$$\alpha(x) = (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_n).$$

(2) Write $f \circ \alpha^{-1}(u, v) = (u, g(u, v))$ on a small product of open balls. Show from the rank assumption that $\mathrm{d}_v g = 0$, and hence $g(u, v) = h(u)$.

(3) Use $\beta(y, z) = (y, z - h(y))$ to finish. Treat $r = 0$ separately by showing that f is constant on a small ball.

A slightly more general version for C^1 maps is given in [7, Theorem 9.32]. The same argument works for C^k maps with $k \geq 1$, giving C^k coordinate changes. For the formulation using linear projections in the 2025 notes, see [2, Chapter 1, "The Rank Theorem"].

3.7. A **retraction** of an open set $U \subseteq \mathbb{R}^n$ onto $A \subseteq U$ is a map $r : U \to A$ with $r|_A = \mathrm{Id}_A$. Viewed as a map into U, it is **idempotent**: $r \circ r = r$.

(1) If $M \subseteq \mathbb{R}^n$ is a manifold and $p \in M$, use a straightening chart, restricted so that its image is a product $A \times B$ with $0 \in B$, to construct a smooth retraction of an open neighborhood $U \subseteq \mathbb{R}^n$ of p onto $M \cap U$. Hint: conjugate $(u, v) \mapsto (u, 0)$ by the chart.

(2) Conversely, suppose $r : U \to U$ is smooth and idempotent. Show that $r(U) = \{x \in U \mid r(x) = x\}$, and, at every fixed point p, $\mathrm{d}r(p)^2 = \mathrm{d}r(p)$. Deduce that its rank equals its trace. Prove that each fixed point p has an open neighborhood $W_0 \subseteq U$ such that $\operatorname{rank} \mathrm{d}r(q) = \operatorname{rank} \mathrm{d}r(p)$ for every fixed point $q \in W_0$.

(3) Fix $p \in r(U)$ and put $d = \operatorname{rank} \mathrm{d}r(p)$. Show that there is an open neighborhood $W \subseteq U$ of p on which $\operatorname{rank} \mathrm{d}r(x) = d$ for every $x \in W$. Hint: choose an open neighborhood $W_1 \subseteq U$ of p on which the rank is at least d, and take $W = W_1 \cap W_0 \cap r^{-1}(W_0)$. Use

$$\mathrm{d}r(r(x))\,\mathrm{d}r(x) = \mathrm{d}r(x), \qquad x \in U,$$

for the reverse inequality.

(4) Apply Exercise 3.6 to find an open neighborhood $V \subseteq U$ of p such that $r(U) \cap V$ is a d-dimensional manifold. Explain the necessary restriction of the target neighborhood: if O is the source neighborhood in the rank theorem, choose $V \subseteq O$ in its target neighborhood so that the coordinate slice in V is contained in $r(O)$. Every fixed point in V is its own image. Conclude that images of smooth idempotents are locally manifolds, although their dimensions may differ between components.

A global theorem, proved using the so-called *tubular neighborhoods*, says that every embedded manifold $M \subseteq \mathbb{R}^n$ is a smooth retract of some open neighborhood of M. In fact, for every open set $O \subseteq \mathbb{R}^n$ containing M, there is a tubular neighborhood V with $M \subseteq V \subseteq O$. This is a more sophisticated theorem in differential topology. Its development uses vector fields and flows, that is, ordinary differential equations formulated on manifolds; see [1, Chapter II, Sections 8 and 11, especially Theorem 11.14 and the paragraph following it]. It is not required in this course.

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