Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD01—Solutions

Throughout, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. All closures and supports are taken in the stated domain.

These solutions follow the corrected statements in the first two sections of Chapter 1. Exercise numbers match those in the notes. In particular, the algebra axioms include scalar compatibility, the power-series formulas use r^α and $\beta!/(\beta-\alpha)!$, and the final exercise requires $g(I) \subsetneq I^2$.

1.1 Continuously differentiable functions of higher order

Exercise 1.1.1: the algebra structure of $C^k(\Omega)$

For $k = 0$, sums and scalar multiples of continuous functions are continuous, so $C^0(\Omega)$ is a vector space over $\mathbb{K}$. Products of continuous functions are continuous as well.

Suppose now that the assertions hold for $C^{k-1}(\Omega)$, where $k \geq 1$. If $f, g \in C^k(\Omega)$ and $c \in \mathbb{K}$, then

$$\partial_i(f+g) = \partial_i f + \partial_i g, \qquad \partial_i(cf) = c\,\partial_i f,$$

and the right-hand sides belong to $C^{k-1}(\Omega)$. Thus $f+g$ and cf belong to $C^k(\Omega)$. The product rule gives

$$\partial_i(fg) = (\partial_i f)g + f(\partial_i g).$$

Every factor on the right belongs to $C^{k-1}(\Omega)$, and the induction hypothesis shows that the right-hand side is in $C^{k-1}(\Omega)$. Hence $fg \in C^k(\Omega)$.

This proves the assertions for every finite k. If f, g are smooth, then they belong to $C^k(\Omega)$ for every finite k; the preceding argument therefore shows that their sums, scalar multiples, and products are smooth. Finally, commutativity, associativity, distributivity, and the identity law follow pointwise from the corresponding properties in $\mathbb{K}$. The scalar-compatibility law holds as well: for every $c \in \mathbb{K}$ and $x \in \Omega$,

$$((cf)g)(x) = (cf(x))g(x) = c(f(x)g(x)) = (f(cg))(x).$$

Consequently, $(cf)g = c(fg) = f(cg)$, and $C^k(\Omega)$ is a commutative $\mathbb{K}$-algebra.

Exercise 1.1.2: composition of C^k maps

We argue by induction on finite k. For $k = 0$, the assertion is the fact that a composition of continuous maps is continuous. Suppose the result holds for $k-1$, and let f and g be of class C^k. Writing $f = (f_1, \ldots, f_m)$ and $g = (g_1, \ldots, g_n)$, the chain rule gives, for $1 \leq q \leq n$ and $1 \leq j \leq l$,

$$\partial_j(g_q \circ f) = \sum_{i=1}^{m} (\partial_i g_q \circ f)\,\partial_j f_i.$$

The induction hypothesis implies that $\partial_i g_q \circ f \in C^{k-1}(U)$. Also $\partial_j f_i \in C^{k-1}(U)$. Exercise 1.1.1 now shows that every term in the sum, and hence every first partial derivative of $g_q \circ f$, belongs to $C^{k-1}(U)$. Therefore $g \circ f$ is of class C^k.

If $k = \infty$, the maps are of class C^r for every finite r, so the finite-order result implies that $g \circ f$ is smooth.

Exercise 1.1.3: polynomials and multi-index notation

(i) Let $\gamma \in \mathbb{N}^n$. Repeated one-variable differentiation shows that for every $\alpha \in \mathbb{N}^n$,

$$\partial^\alpha x^\gamma = \begin{cases} \dfrac{\gamma!}{(\gamma-\alpha)!} x^{\gamma-\alpha}, & \text{if } \alpha \le \gamma, \\ 0, & \text{otherwise.} \end{cases}$$

Thus every partial derivative of every order exists and is continuous. Every monomial is therefore smooth.

(ii) A polynomial is a finite linear combination of monomials. Exercise 1.1.1 and part (i) imply that every polynomial is smooth.

(iii) The one-variable binomial theorem, applied in each coordinate, gives

$$x^\gamma = \prod_{i=1}^{n} \big(a_i + (x_i - a_i)\big)^{\gamma_i} = \sum_{\alpha \le \gamma} \binom{\gamma}{\alpha} a^{\gamma-\alpha} (x-a)^\alpha.$$

For $\alpha \le \gamma$,

$$\frac{1}{\alpha!}(\partial^\alpha x^\gamma)(a) = \frac{1}{\alpha!}\frac{\gamma!}{(\gamma-\alpha)!} a^{\gamma-\alpha} = \binom{\gamma}{\alpha} a^{\gamma-\alpha}.$$

Consequently,

$$x^\gamma = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!}(\partial^\alpha x^\gamma)(a)(x-a)^\alpha.$$

Only the terms with $\alpha \le \gamma$ are nonzero. Taking finite linear combinations yields, for every polynomial f,

$$f(x) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!}(\partial^\alpha f)(a)(x-a)^\alpha.$$

Separating the term $\alpha = 0$ gives the second displayed identity in the exercise. Since sufficiently high derivatives of a polynomial vanish, both sums are essentially finite.

Exercise 1.1.4: power series of several variables

For $\alpha \in \mathbb{N}^n$, write $r^\alpha = \prod_{i=1}^n r_i^{\alpha_i}$.

(i) For each i, let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the i-th coordinate projection. Then

$$\Delta(a,r) = \bigcap_{i=1}^{n} \pi_i^{-1}\big((a_i - r_i, a_i + r_i)\big).$$

Each coordinate projection is continuous, so every set in this finite intersection is open. Hence $\Delta(a,r)$ is open.

(ii) Choose an enumeration of $\mathbb{N}^n$. By the hypothesis the resulting series converges, so its terms tend to zero and are bounded. We may therefore choose a strictly positive constant

$$M \coloneqq 1 + \sup_{\alpha \in \mathbb{N}^n} |c_\alpha t^\alpha| < \infty.$$

Since $|t_i| = r_i$, every α satisfies

$$|c_\alpha| r^\alpha = |c_\alpha t^\alpha| \le M.$$

Thus

$$|c_\alpha| \le \frac{M}{r^\alpha}.$$

(iii) Fix $x \in \Delta(a, r)$, and set

$$q_i \coloneqq \frac{|x_i - a_i|}{r_i} < 1.$$

The estimate in part (ii) gives

$$\sum_{\alpha \in \mathbb{N}^n} |c_\alpha (x-a)^\alpha| \leq M \sum_{\alpha \in \mathbb{N}^n} q^\alpha = M \prod_{i=1}^{n} \frac{1}{1-q_i} < \infty.$$

The series therefore converges absolutely, hence unconditionally, to a scalar which we denote by $f(x)$.

(iv) Choose $0 < \rho_i < r_i$ for every i, and consider the compact polydisk defined by $|x_i - a_i| \leq \rho_i$. Put $q_i = \rho_i / r_i$. For a multi-index γ, formal differentiation gives terms indexed by $\beta \geq \gamma$:

$$\partial^\gamma \left(c_\beta (x-a)^\beta \right) = \frac{\beta!}{(\beta - \gamma)!} c_\beta (x-a)^{\beta - \gamma}.$$

Using part (ii), their absolute values on the compact polydisk are bounded by

$$\frac{M}{r^\gamma} \frac{\beta!}{(\beta - \gamma)!} q^{\beta - \gamma}.$$

After writing $\beta = \gamma + \delta$, the sum of these majorants factors into one-variable series:

$$\frac{M}{r^\gamma} \prod_{i=1}^{n} \left(\sum_{d=0}^{\infty} \frac{(d + \gamma_i)!}{d!} q_i^d \right) = \frac{M \gamma!}{r^\gamma} \prod_{i=1}^{n} \frac{1}{(1-q_i)^{\gamma_i + 1}},$$

which is finite. The Weierstrass M-test therefore gives locally uniform convergence of the series obtained after any number of termwise differentiations. To justify termwise differentiation, apply the fundamental theorem of calculus along coordinate segments to finite partial sums, then pass to the limit using uniform convergence of both the functions and their first derivatives on each smaller polydisk. Repeating this argument for the derivative series proves that f is smooth and

$$(\partial^\gamma f)(x) = \sum_{\beta \geq \gamma} \frac{\beta!}{(\beta - \gamma)!} c_\beta (x-a)^{\beta - \gamma}.$$

At $x = a$, only the term $\beta = \gamma$ remains, and hence

$$(\partial^\gamma f)(a) = \gamma! c_\gamma.$$

1.2 Smooth functions on arbitrary sets and partitions of unity

We use the partition-of-unity theorem on an open neighborhood U of the set under consideration. Its functions are real-valued, take values in $[0,1]$ on U, and have locally finite supports subordinate to the given open cover. The local-to-global extension theorem preserves the given C^k regularity; smoothness is asserted when $k = \infty$.

Exercise 1.2.1: support of a function

Let

$$O \coloneqq \bigcup \{V \subseteq X : V \text{ is open and } f|_V = 0\}.$$

The function f vanishes on O, so O is the largest open set on which f vanishes. Thus the support exists and equals $X \setminus O$.

Set $S = \{x \in X : f(x) \neq 0\}$. Since f vanishes on O, we have $S \subseteq X \setminus O$. The latter set is closed, so $\overline{S} \subseteq X \setminus O$. Conversely, if $x \notin \overline{S}$, then some open neighborhood V of x is disjoint from S. Hence $f|_V = 0$, so $x \in O$. This proves

$$\operatorname{supp}(f) = X \setminus O = \overline{S}.$$

Because $\overline{S}$ is the smallest closed set containing S, the last assertion follows as well.

Exercise 1.2.2: independence of the ambient Euclidean space

Regard $\mathbb{R}^m$ as the coordinate subspace $\mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to \mathbb{R}^m$ be projection onto the first m coordinates.

(i) Suppose first that $U_a \subseteq \mathbb{R}^m$ is open and $g_a \in C^k(U_a)$ agrees with f on $U_a \cap A$. Set

$$V_a \coloneqq \pi^{-1}(U_a) \quad \text{and} \quad h_a \coloneqq g_a \circ \pi.$$

Then V_a is an open neighborhood of a in $\mathbb{R}^n$, $h_a \in C^k(V_a)$, and $h_a = f$ on $V_a \cap A$.

Conversely, suppose that $V_a \subseteq \mathbb{R}^n$ is open and $h_a \in C^k(V_a)$ agrees with f on $V_a \cap A$. The set

$$U_a \coloneqq V_a \cap \mathbb{R}^m$$

is an open neighborhood of a in the subspace $\mathbb{R}^m$, and $g_a \coloneqq h_a|_{U_a}$ is of class C^k. It agrees with f on $U_a \cap A$. This proves the equivalence.

(ii) The same constructions work globally. Given an open neighborhood $U \subseteq \mathbb{R}^m$ of A and an extension $g \in C^k(U)$, set

$$V \coloneqq \pi^{-1}(U) \quad \text{and} \quad h \coloneqq g \circ \pi.$$

Then V is an open neighborhood of A in $\mathbb{R}^n$, and $h|_A = f$. Conversely, from $V \subseteq \mathbb{R}^n$ and $h \in C^k(V)$, take

$$U \coloneqq V \cap \mathbb{R}^m \quad \text{and} \quad g \coloneqq h|_U.$$

Then U is open in $\mathbb{R}^m$, $g \in C^k(U)$, and $g|_A = f$. Thus both the local and global notions are independent of the chosen ambient Euclidean space.

Exercise 1.2.3: local finiteness

(i) For any family of sets,

$$\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}\subseteq\overline{\bigcup_{\lambda\in\Lambda}A_\lambda}.$$

For the reverse inclusion, suppose $x\notin\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}$. By local finiteness, there is an open neighborhood U of x which meets only finitely many sets, say $A_{\lambda_1},\dots,A_{\lambda_r}$. For each j, because $x\notin\overline{A_{\lambda_j}}$, choose an open neighborhood U_j of x disjoint from A_{λ_j}. Then

$$U\cap U_1\cap\cdots\cap U_r$$

is an open neighborhood of x disjoint from every A_λ. Therefore $x\notin\overline{\bigcup_\lambda A_\lambda}$, proving

$$\overline{\bigcup_{\lambda\in\Lambda}A_\lambda}=\bigcup_{\lambda\in\Lambda}\overline{A_\lambda}.$$

(ii) Let $X=\mathbb{R}$, $\Lambda=(0,1)$, and $A_a=\{a\}$ for $a\in(0,1)$. This family is not locally finite, and

$$\overline{\bigcup_{a\in(0,1)}A_a}=[0,1],\qquad\bigcup_{a\in(0,1)}\overline{A_a}=(0,1).$$

Thus closure and union need not commute without local finiteness.

(iii) Fix $x\in X$. Local finiteness gives an open neighborhood U_x meeting only finitely many supports, indexed by a finite set $\Lambda_x\subseteq\Lambda$. If $\lambda\notin\Lambda_x$, then f_λ vanishes on U_x. Consequently, for $y\in U_x$,

$$\sum_{\lambda\in\Lambda}f_\lambda(y)=\sum_{\lambda\in\Lambda_x}f_\lambda(y).$$

The right-hand side is a finite sum of continuous functions. The family is therefore pointwise summable, and its sum is continuous in a neighborhood of every point. Hence it defines a continuous function on X.

Exercise 1.2.4: smooth approximation on an arbitrary subset

(i) Fix $a\in A$. Continuity of f at a gives a relatively open neighborhood W_a of a in A such that

$$x\in W_a\implies|f(x)-f(a)|<\varepsilon.$$

There is an open set $U_a\subseteq\mathbb{R}^n$ with $W_a=U_a\cap A$, which proves the assertion.

(ii) Let $(\varphi_a)_{a\in A}$ be a partition of unity on A subordinate to $(U_a)_{a\in A}$, and define

$$g\coloneqq\sum_{a\in A}f(a)\varphi_a.$$

The family $(\operatorname{supp}(\varphi_a))_{a\in A}$ is locally finite. Thus, near each point, the displayed sum is a finite sum of smooth functions. It follows that g is a well-defined smooth function on a neighborhood of A, and hence is smooth on A.

(iii) For $x\in A$, only finitely many summands are nonzero. If $\varphi_a(x)\neq0$, then $x\in\operatorname{supp}(\varphi_a)\subseteq U_a$, so $|f(x)-f(a)|<\varepsilon$. Using nonnegativity and the unity condition,

$$\begin{aligned}|f(x)-g(x)|&=\left|\sum_{a\in A}\varphi_a(x)\big(f(x)-f(a)\big)\right|\\&\le\sum_{a\in A}\varphi_a(x)|f(x)-f(a)|\\&<\varepsilon\sum_{a\in A}\varphi_a(x)=\varepsilon.\end{aligned}$$

Exercise 1.2.5: a space-filling curve and smooth approximation

(i) Let $C \subseteq [0,1]$ be the middle-thirds Cantor set. Every $x \in C$ has a unique expansion

$$x = \sum_{j=1}^{\infty} \frac{2a_j}{3^j}, \qquad a_j \in \{0,1\},$$

when only expansions using the digits 0 and 2 are allowed. Define

$$p(x) \coloneqq \left(\sum_{j=1}^{\infty} \frac{a_{2j-1}}{2^j}, \sum_{j=1}^{\infty} \frac{a_{2j}}{2^j} \right).$$

For each positive integer m, fixing the first m digits defines a relatively open and closed subset of C, and these sets form a neighborhood basis: their diameters are at most 3^{-m}. If two points have the same first $2m$ digits, each coordinate of their images under p differs by at most 2^{-m}. Hence $p : C \to [0,1]^2$ is continuous. Given $(u,v) \in [0,1]^2$, choose binary expansions for u and v and interleave their digits to obtain the sequence (a_j). Hence p is surjective.

Each component (u,v) of $[0,1] \setminus C$ has endpoints in C. Set $f = p$ on C, and define

$$f(t) = \frac{v-t}{v-u} p(u) + \frac{t-u}{v-u} p(v), \qquad t \in (u,v).$$

These convex combinations lie in $[0,1]^2$. Continuity inside each gap is immediate. To check continuity at $x \in C$, let $t_j \to x$. Terms in C cause no difficulty because p is continuous. For terms in gaps, if the lengths of their gaps tend to zero, both endpoints tend to x, and the convex-combination formula gives $f(t_j) \to p(x)$. Otherwise a subsequence lies in gaps of lengths bounded below. There are only finitely many such disjoint gaps in $[0,1]$, so a further subsequence lies in one fixed gap and approaches one of its endpoints, where the affine extension is continuous. A subsequence on which convergence to $p(x)$ fails would contradict these two cases. Thus f is continuous at x. Finally, $f(C) = p(C) = [0,1]^2$, so f is surjective.

(ii) Fix a continuous $f : I \to I^2$ and $\varepsilon > 0$. For each $s \in I$, choose an open neighborhood U_s of s in $\mathbb{R}$ such that

$$t \in U_s \cap I \implies \|f(t) - f(s)\| < \varepsilon.$$

Choose a smooth partition of unity $(\varphi_s)_{s \in I}$ on I subordinate to this cover, and set

$$g(t) \coloneqq \sum_{s \in I} \varphi_s(t) f(s).$$

The partition functions are defined on the open set $U = \bigcup_{s \in I} U_s$. Local finiteness makes this sum a smooth map on U, whose restriction to I is the required g. Moreover, the nonzero coefficients are nonnegative and sum to one. Since I^2 is convex, this implies $g(t) \in I^2$. As in Exercise 1.2.4,

$$\|g(t) - f(t)\| \le \sum_{s \in I} \varphi_s(t) \|f(s) - f(t)\| < \varepsilon$$

for every $t \in I$.

It remains to prove that $g(I) \subsetneq I^2$. Because g is smooth on I, it extends smoothly to an open neighborhood of I. Its derivative is therefore bounded on I, so g is Lipschitz there; let $L > 0$ be a Lipschitz constant. Partition I into N intervals of length $1/N$. The image of each interval lies in an axis-parallel square of side length $2L/N$, centered at the image of its left endpoint. The sum of the areas of these N squares is

$$N \left(\frac{2L}{N} \right)^2 = \frac{4L^2}{N}.$$

Choose $N > 4L^2$. Finitely many axis-parallel squares with total area less than one cannot cover the unit square. This elementary fact follows by subdividing along all their horizontal and vertical edges: if the unit square were covered, its area would be at most the sum of the covering areas. Consequently, $g(I) \neq I^2$. Together with the previously proved inclusion, this gives $g(I) \subsetneq I^2$.