Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD1

Throughout, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^+ = \{1, 2, \ldots\}$, and $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$. Exercise numbers agree with Chapter 1 of the lecture notes.

1.1 Continuously differentiable functions of higher order

1.1.1. Prove the following statement.

Let Ω be an open set in $\mathbb{R}^n$ and $k \in \mathbb{N}_\infty$. The following hold:

(1) With respect to the usual pointwise addition of functions and multiplication by scalars, $C^k(\Omega)$ is a vector space over $\mathbb{K}$.

(2) Under pointwise multiplication and the above vector space structure, $C^k(\Omega)$ becomes a commutative algebra over $\mathbb{K}$, i.e. in addition to the usual axioms for a $\mathbb{K}$-vector space, with 1_Ω denoting the constant function on Ω with value $1 \in \mathbb{K}$, the following conditions are satisfied for all $f, g, h \in C^k(\Omega)$:

  (i) $fg = gf$;
  (ii) $(fg)h = f(gh)$;
  (iii) $f(g + h) = fg + fh$ and $(f + g)h = fh + gh$;
  (iv) $(cf)g = c(fg) = f(cg)$ for all $c \in \mathbb{K}$;
  (v) $1f = f1 = f$.

1.1.2. Prove the following statement.

Let $l, m, n \in \mathbb{N}^+$ and $k \in \mathbb{N}_\infty$. Suppose $U \subseteq \mathbb{R}^l$ and $V \subseteq \mathbb{R}^m$ are open and $f : U \to V \subseteq \mathbb{R}^m$ and $g : V \to \mathbb{R}^n$ are of class C^k. Then $g \circ f \in C^k(U, \mathbb{R}^n)$.

1.1.3. (**Polynomials and the multi-index notation**) Let $n \in \mathbb{N}^+$. A **monomial function** on $\mathbb{R}^n$ is a function of the form $(x_1, \ldots, x_n) \mapsto x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where each $\alpha_i \in \mathbb{N}$ and we interpret x_i^0 as $1 \in \mathbb{K}$. To simplify notation, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we set $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We also introduce the following notation:

$$\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!, \qquad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} \qquad \text{and} \qquad |\alpha| = \sum_{i=1}^n \alpha_i.$$

A **polynomial function** on $\mathbb{R}^n$ is a function that is a linear combination of monomial functions.

(i) Show that every monomial function on $\mathbb{R}^n$ is smooth.

(ii) Show that every polynomial function on $\mathbb{R}^n$ is smooth.

(iii) Show that for every polynomial function f on $\mathbb{R}^n$ and every $a \in \mathbb{R}^n$, we have

$$\forall x \in \mathbb{R}^n, \qquad \begin{aligned} f(x) &= \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (\partial^\alpha f)(a)(x-a)^\alpha \\ &= f(a) + \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \frac{1}{\alpha!} (\partial^\alpha f)(a)(x-a)^\alpha, \end{aligned}$$

where the sums are essentially finite in the sense of $\partial^\alpha f \equiv 0$ when $|\alpha|$ is sufficiently large, so there is no problem of convergence in the above equalities.

Hint: first treat the one-variable case, then the monomial function case, and finally the polynomial function case.

1.1.4. (**Power series of several variables**) Let $a \in \mathbb{R}^n$ and $r \in \mathbb{R}^n_{>0}$. The **open polydisk** $\Delta(a, r)$ of center a and radii r is defined by

$$\Delta(a, r) := \left\{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \;\middle|\; |x_i - a_i| < r_i \text{ for each } 1 \leq i \leq n \right\}.$$

Recall the multi-index notation in the previous exercise, and let $c_\alpha \in \mathbb{K}$ for each $\alpha \in \mathbb{N}^n$.

(i) Show that $\Delta(a, r)$ is open in $\mathbb{R}^n$.

(ii) Suppose that there exists $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ such that $|t_i| = r_i$ for each $1 \leq i \leq n$, and that the series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha t^\alpha$$

converges unconditionally, i.e. it converges to the same limit in $\mathbb{K}$ with respect to any order of summation over the countable set $\mathbb{N}^n$. Show that there exists $M > 0$ such that

$$\forall \alpha \in \mathbb{N}^n, \qquad |c_\alpha| \leq \frac{M}{r^\alpha} = \frac{M}{r_1^{\alpha_1} \cdots r_n^{\alpha_n}}.$$

(iii) Under the same assumption as in (ii), show that for any $x \in \Delta(a, r)$, the series

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha (x-a)^\alpha$$

converges unconditionally to some $f(x) \in \mathbb{K}$.

(iv) Let $f : \Delta(a, r) \to \mathbb{K}$ be the function given in (iii) (so we keep the same assumption as in (ii)). Show that f is smooth, that

$$\forall \alpha \in \mathbb{N}^n, \qquad (\partial^\alpha f)(a) = \alpha! c_\alpha$$

and that

$$\forall x \in \Delta(a, r), \qquad (\partial^\alpha f)(x) = \sum_{\beta \geq \alpha} \frac{\beta!}{(\beta - \alpha)!} c_\beta (x-a)^{\beta - \alpha},$$

where the sum is over all $\beta \in \mathbb{N}^n$ such that $\beta_i \geq \alpha_i$ for each $1 \leq i \leq n$.

1.2 Smooth functions on arbitrary sets and partitions of unity

1.2.1. Let (X, d) be a metric space. The **support** of a function $f : X \longrightarrow \mathbb{K}$, denoted by $\operatorname{supp}(f)$, is the complement of the largest open set in X on which f vanishes.

Let $f : X \longrightarrow \mathbb{K}$ be a function on a metric space (X, d). Show that the support of f exists and $\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}}$. Moreover, $\operatorname{supp}(f)$ is the smallest closed set containing $\{x \in X \mid f(x) \neq 0\}$.

1.2.2. Consider $\mathbb{R}^\infty = \{(x_n)_{n\geq 1}\}$, the real vector space of all sequences of real numbers. For each $n \in \mathbb{N}^+$, we identify $\mathbb{R}^n$ with the linear subspace of $\mathbb{R}^\infty$ consisting of all sequences whose terms starting from the $n + 1$-th are all zero, i.e.

$$\mathbb{R}^n := \left\{(x_k)_{k\geq 1} \;\middle|\; \forall k \geq n+1, \quad x_k = 0\right\}.$$

Suppose $m, n \in \mathbb{N}^+$ with $m \leq n$. We then have $\mathbb{R}^m \subseteq \mathbb{R}^n$ according to the above identification. Let $A \subseteq \mathbb{R}^m \subseteq \mathbb{R}^n$, $k \in \mathbb{N}_\infty$, and suppose $f : A \longrightarrow \mathbb{K}$ is a function.

(1) Show that for any $a \in A$, the following are equivalent:

   (i) There exists an open neighborhood U_a of a in $\mathbb{R}^m$ and a $g_a \in C^k(U_a)$ such that $f|_{U_a \cap A} = g_a|_{U_a \cap A}$.

   (ii) There exists an open neighborhood V_a of a in $\mathbb{R}^n$ and an $h_a \in C^k(V_a)$ such that $f|_{V_a \cap A} = h_a|_{V_a \cap A}$.

   Remark: this means the notion of f being of class C^k at $a \in A$ is independent of the choice of d as long as $A \subseteq \mathbb{R}^d$.

(2) Show that the following are equivalent:

   (i) There exists an open neighborhood U of A in $\mathbb{R}^m$, and a $g \in C^k(U)$ such that $g|_A = f$.

   (ii) There exists an open neighborhood V of A in $\mathbb{R}^n$, and a $h \in C^k(V)$ such that $h|_A = f$.

   Remark: this means the notion of f being of class C^k on A is independent of the choice of d as long as $A \subseteq \mathbb{R}^d$.

1.2.3. (**Local finiteness**) Let (X, d) be a metric space, and $(A_\lambda)_{\lambda\in\Lambda}$ an indexed family of subsets of X. We say $(A_\lambda)_{\lambda\in\Lambda}$ is **locally finite** on X if for each $x \in X$, there exists an open neighborhood U_x of x in X such that

$$\{\lambda \in \Lambda \mid U_x \cap A_\lambda \neq \emptyset\}$$

is finite.

(1) Show that if $(A_\lambda)_{\lambda\in\Lambda}$ is locally finite on X, then taking closures commutes with taking unions of $(A_\lambda)_{\lambda\in\Lambda}$, i.e.

$$\overline{\bigcup_{\lambda\in\Lambda} A_\lambda} = \bigcup_{\lambda\in\Lambda} \overline{A_\lambda}.$$

(2) Dropping the local finiteness assumption, give a counterexample that demonstrates the failure of commutation of taking closures and taking unions.

   Hint: one possible example goes as follows: take any non-empty subset $A \subseteq \mathbb{R}$ that is not closed, and consider the indexed family $(\{a\})_{a\in A}$.

(3) Show that if $(f_\lambda)_{\lambda\in\Lambda}$ is a family of continuous functions from X to $\mathbb{K}$ such that $(\operatorname{supp} f_\lambda)_{\lambda\in\Lambda}$ is locally finite, then

$$f := \sum_{\lambda\in\Lambda} f_\lambda$$

   is a well-defined *continuous* function from X to $\mathbb{K}$.

1.2.4. Let $n \in \mathbb{N}^+$ and $A \subseteq \mathbb{R}^n$. The goal of this exercise is to show that for any continuous function $f : A \to \mathbb{K}$ and any $\varepsilon > 0$, there exists a smooth function g on A such that $|f(a) - g(a)| < \varepsilon$ for any $a \in A$.

(1) Show that for every $a \in A$, there exists an open neighborhood U_a of a in $\mathbb{R}^n$ such that for any $x \in U_a \cap A$, we have $|f(x) - f(a)| < \varepsilon$.

(2) Consider the open cover $\mathcal{U} \coloneqq (U_a)_{a\in A}$ of A and let $(\varphi_a)_{a\in A}$ be a partition of unity on A subordinate to $\mathcal{U}$. Show that

$$g \coloneqq \sum_{a\in A} f(a)\varphi_a$$

is a well-defined smooth function on A.

(3) Show that $|f(a) - g(a)| < \varepsilon$ for every $a \in A$.

1.2.5. This exercise, while not particularly difficult, may involve some additional reading. Let $I = [0, 1]$ be the unit interval, and $I^2 \subseteq \mathbb{R}^2$ the unit cube.

(1) Show that there exists a continuous map $f : I \to I^2$ that is surjective.

Hint: you may search for "the Peano curve" if you get stuck.

(2) Show, using partitions of unity, that for any continuous map $f : I \to I^2$ and any $\varepsilon > 0$, there exists a smooth map $g : I \to \mathbb{R}^2$ with $g(I) \subseteq I^2$ such that $|g(t) - f(t)| < \varepsilon$ for every $t \in I$, and $g(I) \subsetneq I^2$.

Remark: there is a general result called Sard's lemma, which implies that there is no surjective smooth map from I onto I^2.