Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD02 — Solutions

Complete solutions to TD02. The exercise and part numbers agree with the sheet. Smoothness on subsets is understood in the extension sense.

1.3 Manifolds: charts, zeros and graphs

1.3.1. (**Zero-dimensional manifolds**) The only nonempty open subset of $\mathbb{R}^0$ is $\{0\}$. A chart at p therefore has domain $\{p\}$, which must be relatively open in M. This is exactly the stated discreteness condition. Conversely, if every singleton is relatively open, the maps $\{p\} \to \mathbb{R}^0$ and $\mathbb{R}^0 \to \{p\}$ are inverse smooth maps: the latter is constant, and the former is the restriction of the unique map to $\mathbb{R}^0$. They give a chart at every point. The nonemptiness assumption excludes the empty manifold, whose dimension is -1 by convention.

1.3.2. (**Torus and open Möbius band**)

(1) Put $\rho = \sqrt{x^2 + y^2}$. If a point of T had $\rho = 0$, its equation would imply $z^2 + R^2 - r^2 = 0$, impossible since $R > r$. Also $\rho^2 + z^2 + R^2 - r^2 > 0$, so taking the positive square root of the defining equation gives

$$\rho^2 + z^2 + R^2 - r^2 = 2R\rho \quad \Longleftrightarrow \quad (\rho - R)^2 + z^2 = r^2.$$

On the open set $\rho > 0$, the function $h(x, y, z) = (\rho - R)^2 + z^2 - r^2$ is smooth and

$$\operatorname{grad} h = \left(\frac{2(\rho - R)x}{\rho}, \frac{2(\rho - R)y}{\rho}, 2z \right).$$

A zero of this gradient would have $z = 0$ and $\rho = R$, and hence $h = -r^2 \neq 0$. Thus T is a regular level set of codimension one. It is closed by its polynomial equation and bounded because $\rho \leq R + r$ and $|z| \leq r$. It is nonempty, for instance containing $(R + r, 0, 0)$, and is a compact two-dimensional manifold.

(2) The identities $\cos(\theta/2 + \pi) = -\cos(\theta/2)$ and $\sin(\theta/2 + \pi) = -\sin(\theta/2)$ give $p(\theta + 2\pi, t) = p(\theta, -t)$. Since $R + t\cos(\theta/2) > R - a > 0$, the polar angle of the first two coordinates of $p(\theta, t)$ is θ modulo 2π, and their radius is $\rho = R + t\cos(\theta/2)$.

Let $S_J \subseteq \mathbb{R}^3$ be the open set on which $\rho > 0$ and the polar angle has a representative in J. Because the length of J is less than 2π, this representative θ_J is unique and smooth. The formula

$$q(x, y, z) = \big(\theta_J(x, y), (\rho - R)\cos(\theta_J(x, y)/2) + z\sin(\theta_J(x, y)/2)\big)$$

defines a smooth map on S_J. Direct substitution gives $q(p(\theta, t)) = (\theta, t)$ for $(\theta, t) \in J \times (-a, a)$. Conversely, every point of $M \cap S_J$ can be represented using an angle in J: subtract a multiple of 2π and change the sign of t if necessary. Thus

$$p(J \times (-a, a)) = M \cap S_J, \qquad p \circ q = \mathrm{Id}_{M \cap S_J}.$$

The restriction is a diffeomorphism onto a relatively open subset. These subsets cover M, proving the assertion.

1.3.3. (**Singular equations**)

(1) The zero set X is the union of the two coordinate axes. Suppose a relatively open neighborhood of zero in X were a manifold of dimension d. It contains nonzero points, at which the set is locally an open interval, so uniqueness of dimension gives $d = 1$. Choose a chart at zero whose coordinate image is an open interval. Its domain contains points of all four half-axes. After deleting zero, no connected subset can meet two different half-axes: they are the four connected components of $X \setminus \{0\}$. The punctured chart therefore has at least four components. An interval with one interior point removed has exactly two, a contradiction.

(2) The map $t \mapsto (t^2, t^3)$ is continuous and injective. For $(x, y) \in C$, we have $x \geq 0$ and $x = (\sqrt[3]{y})^2$, so it is surjective with continuous inverse $(x, y) \mapsto \sqrt[3]{y}$. Away from zero, $y^2 - x^3$ has nonzero gradient on C, so any manifold neighborhood of zero would have dimension one.

For a smooth curve $\gamma(s) = (x(s), y(s))$ in C through zero, $x(s) \geq 0 = x(0)$ implies $x'(0) = 0$. Thus $x(s) = o(|s|)$, and

$$\frac{|y(s)|}{|s|} = \left(\frac{x(s)}{|s|}\right)^{3/2} |s|^{1/2} \longrightarrow 0.$$

Hence $\gamma'(0) = 0$. But a local parametrization f with smooth inverse extension g satisfies $\mathrm{d}g(f(0))f'(0) = 1$. It cannot have zero derivative, giving the contradiction.

(3) For $f(x, y) = y^2$, $\mathrm{d}f(x, y) = (0, 2y)$ vanishes on the zero set $\mathbb{R} \times \{0\}$. That set is diffeomorphic to $\mathbb{R}$ by coordinate projection. The alternative defining function $g(x, y) = y$ is a submersion. Thus the regularity condition concerns the chosen equation, not only its zero set.

1.3.4. (**Simplex and ball**)

(1) Translate K so that zero is an interior point, and choose r, R as in the question. Its gauge satisfies

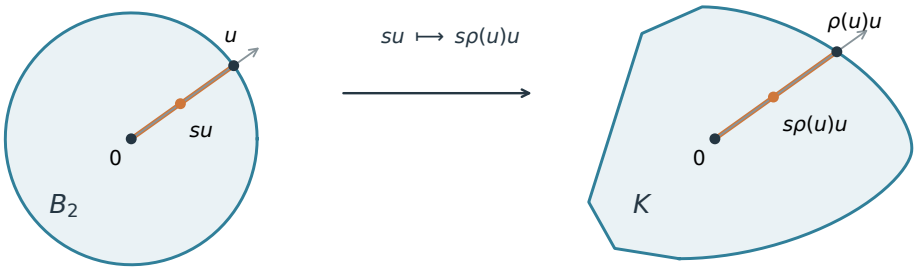
$$\frac{|x|}{R} \leq p(x) \leq \frac{|x|}{r}, \qquad p(\lambda x) = \lambda p(x) \quad (\lambda \geq 0).$$

The bounds follow from the ball inclusions, and positive homogeneity follows by rescaling the defining infimum; $p(0) = 0$. If $a > p(x)$ and $b > p(y)$, then $x \in aK$ and $y \in bK$. Indeed, membership holds at some smaller parameter, and convexity with $0 \in K$ allows an increase of that parameter. Convexity now gives

$$\frac{x + y}{a + b} = \frac{a}{a + b}\frac{x}{a} + \frac{b}{a + b}\frac{y}{b} \in K.$$

Taking infima proves $p(x + y) \leq p(x) + p(y)$. Apply this inequality in both orders to obtain $|p(x) - p(y)| \leq |x - y|/r$.

Closedness of K shows $K = \{x \mid p(x) \leq 1\}$: for $x \neq 0$, take positive parameters decreasing to $p(x) > 0$, obtaining $x/p(x) \in K$; then use convexity. On S^{n-1}, the function $\rho(u) = 1/p(u)$ is continuous with $r \leq \rho(u) \leq R$. Every ray meets K in the segment $\{tu \mid 0 \leq t \leq \rho(u)\}$. Consequently the map $H(0) = 0$, $H(su) = s\rho(u)u$ for $0 < s \leq 1$, is a continuous bijection $B_n \to K$. Continuity at zero follows from $|H(su)| \leq Rs$. A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.



On each ray, the map in (1) sends the segment from 0 to u onto the segment from 0 to $\rho(u)u$, preserving the fraction s of its length. Here the boundary of K consists of straight segments and a curved arc of varying curvature.

(2) The affine map a in the question has inverse $(x_0, \dots, x_n) \mapsto (x_1, \dots, x_n)$ on H. Both maps extend smoothly to their ambient spaces, and the inequalities describing K correspond exactly to the nonnegative coordinates of Δ_n. The set K is compact and convex and has nonempty interior, for example containing a small ball about $(1/(2n), \dots, 1/(2n))$. Part (1) gives the claimed homeomorphism.

(3) Write $f = \varphi \circ a : K \to B_n$ and $g = f^{-1}$. Fix $x \in \operatorname{Int} K$, and take smooth extensions $\widetilde{f}, \widetilde{g}$. On an open neighborhood of x contained in $\operatorname{Int} K$, their composition equals the identity. Thus

$$\mathrm{d}\widetilde{g}(f(x))\,\mathrm{d}\widetilde{f}(x) = I_n.$$

The derivative of $\widetilde{f}$ is invertible at x. The inverse function theorem supplies an open neighborhood of $f(x)$ contained in $f(\operatorname{Int} K) \subseteq B_n$. Hence $f(x) \in \operatorname{Int} B_n$. Apply the same argument to g to obtain equality of interiors. Taking complements inside the two closed sets proves equality of boundaries. Restrictions of the smooth maps and their inverses remain smooth.

(4) At $\gamma(0) = e_i$, every coordinate γ_j with $j \neq i$ is nonnegative and zero at zero. Therefore $\gamma_j'(0) = 0$. Differentiating $\sum_{j=0}^{n} \gamma_j(s) = 1$ also gives $\gamma_i'(0) = 0$. If $\psi : \partial\Delta_n \to S^{n-1}$ were a diffeomorphism, put $q = \psi(e_i)$ and choose a unit vector $v \perp q$; this is possible since $n \geq 2$. The curve $c(s) = (\cos s)q + (\sin s)v$ lies in the sphere, starts at q, and has derivative $v \neq 0$. The smooth curve $\psi^{-1} \circ c$ has derivative zero at zero. Composing it with an ambient extension of ψ and applying the chain rule would give $c'(0) = 0$, a contradiction. Part (3) rules out a diffeomorphism of the closed sets. For $n = 1$, both sets are closed intervals and $(x_0, x_1) \mapsto 2x_1 - 1$ is a diffeomorphism $\Delta_1 \to B_1$.

(5) The exponential formula has positive coordinates summing to one. Its ratios $x_i/x_0 = e^{t_i}$ show that the proposed logarithmic map is its inverse. The logarithms extend smoothly to the ambient open set on which all coordinates are positive, so this is a diffeomorphism in the extension sense. For $b(t) = t/\sqrt{1 + |t|^2}$, we have $|b(t)|^2 = |t|^2/(1 + |t|^2) < 1$. Solving for t gives the stated smooth inverse on the open ball. Composing these diffeomorphisms proves the assertion.

1.3.5. (Classification of smooth manifolds in dimension one)

(1) For a local parametrization f, a smooth ambient extension g of its inverse gives $\mathrm{d}g(f(t))f'(t) = 1$, hence $f'(t) \neq 0$. The arclength function is smooth with $s'(t) = |f'(t)| > 0$, so it is a diffeomorphism onto an open interval. The chain rule shows that $f \circ s^{-1}$ has unit speed. If f and h have unit speed and $h = f \circ \eta$, then $1 = |h'| = |\eta'|$. Continuity makes η' constantly 1 or constantly -1 on each connected overlap. Integration gives $\eta(s) = s + c$ or $\eta(s) = -s + c$.

(2) At a point, the possible unit velocities are $f'(s)$ and $-f'(s)$ for any unit speed chart centered there; part (1) makes this pair independent of the chart. Fix one velocity at time zero. In an arclength

coordinate, a curve of unit speed satisfies $s'(t) = 1$ or $s'(t) = -1$, with the sign determined by its initial velocity. Thus the initial data determine the curve locally. Two curves with these data agree on their common interval: agreement of their points and velocities is both open by this local observation and closed by continuity. Now take the union of all their domains. This is an open interval I containing zero. The curves agree on overlaps, so they define a smooth curve $\gamma : I \to M$ of unit speed. By construction it is the unique maximal curve with the prescribed initial data.

(3) The same coordinate description shows that γ is a local diffeomorphism, so its image is open. Suppose p is in the closure of its image, and choose a unit speed chart $f : (-2\varepsilon, 2\varepsilon) \to M$ with $f(0) = p$. There are $t_0 \in I$ and $s_0 \in (-\varepsilon, \varepsilon)$ with $\gamma(t_0) = f(s_0)$. For the sign $\delta \in \{-1, 1\}$ determined by the velocity, define

$$\eta(t) = f(s_0 + \delta(t - t_0)), \qquad J = \{t \mid |s_0 + \delta(t - t_0)| < 2\varepsilon\}.$$

Local uniqueness and the open-and-closed argument give $\eta = \gamma$ on the interval $I \cap J$. Their union is therefore an extension to $I \cup J$. Maximality implies $J \subseteq I$, in particular $t_0 - \delta s_0 \in I$, at which γ equals p. The image is closed as well as open, and connectedness of M gives surjectivity.

(4) At a repeated point, the two unit velocities agree or are opposite. In the latter case, the curves $u \mapsto \gamma(s + u)$ and $u \mapsto \gamma(t - u)$ have the same initial data at zero. Uniqueness gives their equality throughout a neighborhood of $[0, (t - s)/2]$. Differentiating at the midpoint gives $\gamma'((s + t)/2) = -\gamma'((s + t)/2)$, impossible. Thus the velocities agree. Uniqueness also gives $\gamma(s+u) = \gamma(t+u)$ for all sufficiently small $|u|$. Repeating the segment from s to t therefore defines a smooth periodic curve on all of $\mathbb{R}$: this equality ensures smooth agreement across every joining point. It agrees with γ on their common interval by uniqueness. Maximality forces $I = \mathbb{R}$.

Let H be the additive group of periods. It is closed: if $h_j \to h$ and $h_j \in H$, continuity gives $\gamma(u + h) = \gamma(u)$ for every u. Local injectivity at zero gives $\varepsilon > 0$ with $H \cap (-\varepsilon, \varepsilon) = \{0\}$. Since H contains the positive period $t - s$, the number $L = \inf(H \cap (0, \infty))$ is positive and belongs to H by closedness. Division by L leaves a remainder in $H \cap [0, L)$, necessarily zero. Hence $H = L\mathbb{Z}$. Conversely, any equality $\gamma(u) = \gamma(v)$ gives equal velocities and, by uniqueness on $\mathbb{R}$, makes $v - u$ a period. The fibers are exactly the cosets of $L\mathbb{Z}$.

Thus the displayed map from S^1 to M is a bijection. On every sufficiently short angle interval it is a composition of local parametrizations, with a smooth inverse. These local descriptions prove continuity, smoothness, and smoothness of the inverse; the local-to-global extension theorem from the notes applies to both maps. It is a diffeomorphism.

(5) In the injective case, surjectivity and the local diffeomorphism property make $\gamma : I \to M$ a diffeomorphism: the local inverses agree to form its smooth inverse. A finite interval (a, b) is diffeomorphic to $\mathbb{R}$ by $t \mapsto \log((t - a)/(b - t))$. For (a, ∞) use $\log(t - a)$, and for $(-\infty, b)$ use $-\log(b - t)$; for $\mathbb{R}$ use the identity. Compactness distinguishes S^1 from $\mathbb{R}$, proving the claimed classification.

Finally, each point of any one-dimensional manifold has a connected open interval neighborhood, contained in its connected component. Components are therefore open. A subspace of $\mathbb{R}^n$ has a countable base. Choose a nonempty basic open set contained in each component; disjoint components require different basic sets. There are at most countably many components, and the connected classification applies to each.

1.3.6. (**Constant rank theorem**)

(1) After translations assume $a = 0$ and $f(a) = 0$. For $r > 0$, rank r gives an invertible r-by-r minor. Permute coordinates to place it in the first r rows and columns. The derivative at zero of

$$\alpha(x) = (f_1(x), \ldots, f_r(x), x_{r+1}, \ldots, x_n)$$

is block upper triangular, with that minor and I_{n-r} on the diagonal. It is invertible. By the inverse function theorem, restrict to an open neighborhood $U \subseteq \Omega_0$ on which α is a diffeomorphism onto a product $A \times B$ of open balls centered at zero.

(2) In these coordinates, $f \circ \alpha^{-1}(u, v) = (u, g(u, v))$, and

$$\mathrm{d}(f \circ \alpha^{-1}) = \begin{pmatrix} I_r & 0 \\ \mathrm{d}_u g & \mathrm{d}_v g \end{pmatrix}.$$

Subtracting appropriate multiples of the first r rows shows that its rank is $r + \operatorname{rank} \mathrm{d}_v g$. The rank is r by the chain rule, so $\mathrm{d}_v g = 0$. For fixed u, integrate along line segments in the convex ball B to obtain $g(u, v) = g(u, 0)$. Put $h(u) = g(u, 0)$, a smooth map on A.

(3) On the open neighborhood $V = A \times \mathbb{R}^{m-r}$ of zero define $\beta(y, z) = (y, z - h(y))$. It is a diffeomorphism of V onto itself, with inverse $(y, w) \mapsto (y, w + h(y))$. Since $h(0) = 0$, it fixes zero, and $f(U) \subseteq V$. For every $(u, v) \in A \times B$, the composite equals $(u, 0)$. Undoing the original translations and permutations gives the required maps and neighborhoods. If $r = 0$, then $\mathrm{d}f = 0$ on an open ball about a contained in Ω_0. Integration along line segments shows that $f = f(a)$ there. Source and target translations give the asserted zero normal form. The arguments also include $r = n$ or $r = m$, using $\mathbb{R}^0$ for a missing coordinate block.

1.3.7. (**Smooth retractions**)

(1) Choose a straightening chart $\Phi : U \to A \times B$ with $0 \in B$ and $\Phi(M \cap U) = A \times \{0\}$. Then

$$r = \Phi^{-1} \circ \pi \circ \Phi, \qquad \pi(u, v) = (u, 0),$$

is a smooth map $U \to U$ with image $M \cap U$, and it fixes every point of this image. Thus it is the desired retraction and is idempotent.

(2) If $y = r(x)$, then $r(y) = r(r(x)) = r(x) = y$; conversely, a fixed point is its own image. Differentiating idempotence gives $\mathrm{d}r(r(x))\,\mathrm{d}r(x) = \mathrm{d}r(x)$. At a fixed point p, put $P = \mathrm{d}r(p)$; then $P^2 = P$. Every vector decomposes as $v = Pv + (v - Pv)$, with summands in $\operatorname{im} P$ and $\ker P$, respectively. Their intersection is zero. In a basis adapted to this direct sum, P is $\operatorname{diag}(I_d, 0)$, so its rank and trace both equal d.

By continuity of $x \mapsto \operatorname{Tr} \mathrm{d}r(x)$, choose an open neighborhood $W_0 \subseteq U$ of p such that

$$|\operatorname{Tr} \mathrm{d}r(x) - d| < 1/2, \qquad x \in W_0.$$

At fixed points this trace is an integer equal to the rank, and therefore equals d.

(3) If $d > 0$, select a nonzero d-by-d minor of $\mathrm{d}r(p)$. Its determinant remains nonzero on an open neighborhood $W_1 \subseteq U$ of p, giving rank at least d. If $d = 0$, take $W_1 = U$. The set $W = W_1 \cap W_0 \cap r^{-1}(W_0)$ is an open neighborhood of p. For $x \in W$, the point $r(x) \in W_0$ is fixed, so part (2) gives $\operatorname{rank} \mathrm{d}r(r(x)) = d$. The differentiated idempotence identity implies $\operatorname{im} \mathrm{d}r(x) \subseteq \operatorname{im} \mathrm{d}r(r(x))$, giving the reverse rank inequality.

(4) Apply the constant rank theorem to r at p, restricting its source neighborhood $O \subseteq W$ so that its source coordinates form a product $A \times B$ centered at zero. Let $Q \subseteq U$ be a target neighborhood with coordinates β. The normal form gives

$$\beta(r(O)) = A \times \{0\}.$$

Choose an open neighborhood $V \subseteq Q \cap O$ of p such that the first d coordinates of $\beta(V)$ lie in A. Such a choice is possible by continuity of the coordinate projection. For $x \in V$, if $r(x) = x$

then $x \in r(O)$, since $x \in O$. Conversely, if $\beta(x) \in \mathbb{R}^d \times \{0\}$, then its first coordinates lie in A, so $x \in r(O)$ and is fixed. It follows that

$$\beta(r(U) \cap V) = \beta(V) \cap (\mathbb{R}^d \times \{0\}).$$

This is a straightening chart for $r(U) \cap V$, proving it is a d-dimensional manifold. The rank is locally constant on the fixed point set, so it is constant on each connected component, but it need not be the same on different components. For example, on the union of two disjoint open balls, use the identity on one ball and the constant map to the center on the other. The image has components of dimensions n and zero.