Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD02

Smoothness on subsets is understood in the extension sense of the lecture notes. We use $\mathbb{R}^0 = \{0\}$. Exercise numbers agree with Chapter 1 of the lecture notes.

1.3 Manifolds: charts, zeros and graphs

1.3.1. (**Zero-dimensional manifolds**) Show that a nonempty subset $M \subseteq \mathbb{R}^n$ is a zero-dimensional manifold if and only if it is discrete in its subspace topology, that is, every $p \in M$ has an open neighborhood $W \subseteq \mathbb{R}^n$ with $W \cap M = \{p\}$.

1.3.2. (**Torus and open Möbius band**)

(1) Fix $R > r > 0$. Show that

$$T = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; (x^2 + y^2 + z^2 + R^2 - r^2)^2 = 4R^2(x^2 + y^2)\right\}$$

is a compact smooth manifold of dimension two. Hint: first show that $\rho = \sqrt{x^2 + y^2} > 0$ on T, and rewrite the equation as $(\rho - R)^2 + z^2 = r^2$. Check that the gradient of this defining function does not vanish on T.

(2) Fix $R > a > 0$, and define

$$p(\theta, t) = \big((R + t\cos(\theta/2))\cos\theta, (R + t\cos(\theta/2))\sin\theta, t\sin(\theta/2)\big).$$

Let $M = p(\mathbb{R} \times (-a, a))$, the **open Möbius band**. Verify $p(\theta + 2\pi, t) = p(\theta, -t)$. For each θ_0, choose an open interval J containing θ_0 of length less than 2π. Show that the restriction of p to $J \times (-a, a)$ is a diffeomorphism onto a relatively open subset of M. Hint: use a smooth branch $\theta \in J$ of the polar angle and the formula $t = (\rho - R)\cos(\theta/2) + z\sin(\theta/2)$. Deduce that M is a two-dimensional manifold.

The torus with $R = 2$, $r = 0.7$, and the open Möbius band with $R = 2$, $a = 0.65$. The edge of the pictured band is not included.

1.3.3. (**Singular equations**)

(1) Show that the zero set of $f(x,y) = xy$ is not a manifold near the origin. Hint: away from the origin its dimension would have to be one; deleting the origin from a small connected chart would leave two components, whereas the crossing has four branches.

(2) Show that $C = \{(x,y) \in \mathbb{R}^2 \mid y^2 = x^3\}$ is homeomorphic to $\mathbb{R}$ via $t \mapsto (t^2, t^3)$, but is not a smooth manifold at the origin. Hint: for any smooth curve $\gamma(s) = (x(s), y(s))$ in C with $\gamma(0) = 0$, use $x(s) \geq 0$ and $|y(s)| = x(s)^{3/2}$ to show $\gamma'(0) = 0$. Compare with the derivative of a local parametrization.

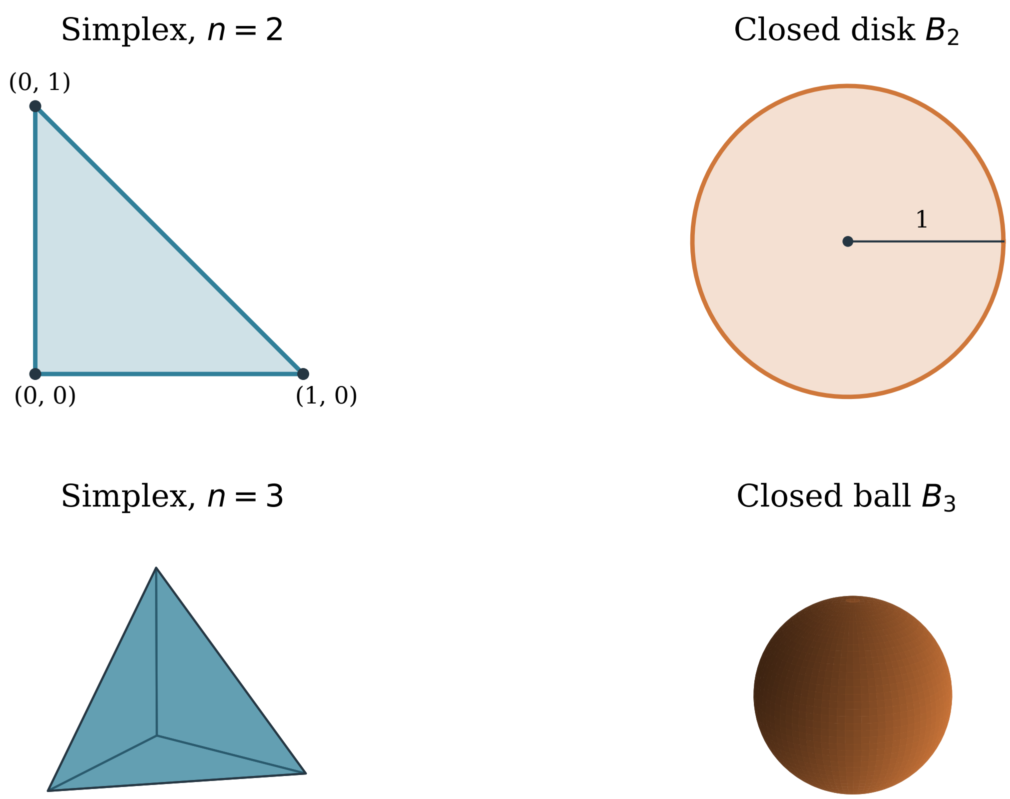
(3) Show that $f(x,y) = y^2$ has derivative zero at every point of its zero set, although that set is a smooth manifold. Explain why regularity of a particular defining equation is sufficient but not necessary.

1.3.4. (**Simplex and ball**) For $n \geq 2$, let

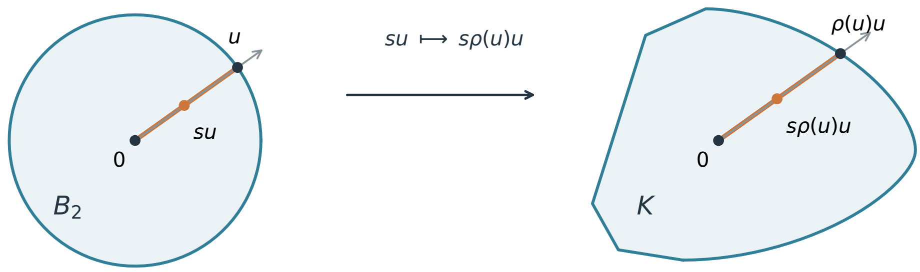
$$\Delta_n = \left\{ x \in \mathbb{R}^{n+1}_{\geq 0} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \right\}, \qquad B_n = \{ y \in \mathbb{R}^n \mid |y| \leq 1 \}.$$

Interiors and boundaries of Δ_n are taken relative to $H = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} x_i = 1 \right\}$.

(1) Show that every compact convex set $K \subseteq \mathbb{R}^n$ with nonempty interior is homeomorphic to B_n. Hint: translate so that $0 \in \operatorname{Int} K$, choose $0 < r < R$ with $rB_n \subseteq K \subseteq RB_n$, and define $p(x) = \inf\{t > 0 \mid x \in tK\}$. Prove subadditivity and $|p(x) - p(y)| \leq |x - y|/r$. The radial function $\rho(u) = 1/p(u)$ on S^{n-1} is continuous; use $su \mapsto s\rho(u)u$, $0 \leq s \leq 1$.



The simplex and closed unit ball in dimensions two and three. The simplices are shown in the coordinates of the affine map a from part (2).



On each ray, the map in (1) sends the segment from 0 to u onto the segment from 0 to $\rho(u)u$, preserving the fraction s of its length. Here the boundary of K consists of straight segments and a curved arc of varying curvature.

(2) The affine map $a(y_1, \dots, y_n) = (1 - \sum_{i=1}^n y_i, y_1, \dots, y_n)$ maps $\mathbb{R}^n$ bijectively onto H. Show that it restricts to a diffeomorphism from $K = \left\{ y \in \mathbb{R}^n_{\geq 0} \;\middle|\; \sum_{i=1}^n y_i \leq 1 \right\}$ to Δ_n, and deduce a homeomorphism $\Delta_n \cong B_n$.

(3) Suppose that $\varphi : \Delta_n \longrightarrow B_n$ is a diffeomorphism. Show that it maps interiors onto interiors, and hence restricts to a diffeomorphism of boundaries. Hint: use a to work in $\mathbb{R}^n$. At an interior point, differentiate the inverse identities for smooth extensions and apply the inverse function theorem. Apply the same argument to the inverse map.

(4) For a vertex e_i of Δ_n, prove that every smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \partial\Delta_n$ with $\gamma(0) = e_i$ satisfies $\gamma'(0) = 0$. Hint: each coordinate other than the i-th has a local minimum, and the sum of the coordinates is constant. Every point of $\partial B_n = S^{n-1}$ lies on a smooth curve with nonzero velocity. Use the chain rule to rule out a diffeomorphism of the boundaries, and hence of the closed sets. Explain why $n = 1$ is an exception.

(5) Show that the interiors are nevertheless diffeomorphic. Verify that

$$(t_1, \dots, t_n) \longmapsto \frac{(1, e^{t_1}, \dots, e^{t_n})}{1 + \sum_{i=1}^n e^{t_i}}$$

is a diffeomorphism from $\mathbb{R}^n$ onto $\operatorname{Int} \Delta_n$, with inverse

$$x \longmapsto (\log(x_1/x_0), \dots, \log(x_n/x_0)).$$

Combine it with the diffeomorphism $t \mapsto t/\sqrt{1+|t|^2}$ from $\mathbb{R}^n$ to $\operatorname{Int} B_n$, whose inverse is $y \mapsto y/\sqrt{1-|y|^2}$.

1.3.5. **(Classification of smooth manifolds in dimension one)** Let $M \subseteq \mathbb{R}^n$ be a nonempty *connected* one-dimensional manifold. A smooth curve is called of **unit speed** if $|\gamma'(t)| = 1$.

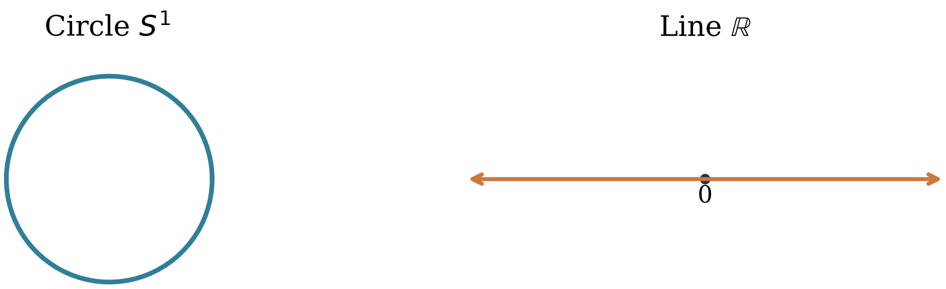
(1) For a local parametrization $f : (a, b) \longrightarrow M$, show that $f'(t) \neq 0$. Reparametrize by $s(t) = \int_{t_0}^t |f'(u)| \, du$ to obtain a unit speed parametrization. Show that the coordinate changes between such parametrizations have the form $s \mapsto s + c$ or $s \mapsto -s + c$ on each connected overlap.

(2) Fix a point and one of its two unit velocities. By joining compatible local parametrizations, construct a maximal unit speed curve $\gamma : I \longrightarrow M$, where I is an open interval containing zero. Prove uniqueness for these initial data. Hint: in a unit speed coordinate a curve has coordinate derivative constantly 1 or constantly -1.

(3) Show that $\gamma(I) = M$. Hint: the image is open. If p is in its closure, choose a unit speed chart $f : (-2\varepsilon, 2\varepsilon) \longrightarrow M$ centered at p, and a point of the image in $f((-\varepsilon, \varepsilon))$. Continue through that chart in both directions; maximality forces the curve to reach p. Thus the image is also closed.

(4) If $\gamma(s) = \gamma(t)$ for $s < t$, show that their velocities agree. Hint: opposite velocities and uniqueness would give $\gamma(s+u) = \gamma(t-u)$ up to the midpoint, contradicting unit speed there. Deduce that $I = \mathbb{R}$ and that γ is periodic. Show that its periods form $L\mathbb{Z}$ for some $L > 0$, using local injectivity and closedness of the period group. Prove that $(\cos(2\pi t/L), \sin(2\pi t/L)) \mapsto \gamma(t)$ is a diffeomorphism from S^1 onto M.

(5) If γ is injective, show that it is a diffeomorphism $I \to M$, and recall that every nonempty open interval is diffeomorphic to $\mathbb{R}$. Conclude that M is diffeomorphic to S^1 when compact, and to $\mathbb{R}$ when noncompact. Finally, show that an arbitrary one-dimensional manifold has at most countably many connected components, each open and of one of these two types.



The two types of nonempty connected one-dimensional manifolds.

1.3.6. (Constant rank theorem) Let $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be smooth, with Ω open. Suppose $a \in \Omega$ has an open neighborhood $\Omega_0 \subseteq \Omega$ such that $\operatorname{rank} df(x) = r$ for every $x \in \Omega_0$. Prove that there are open neighborhoods $U \subseteq \Omega_0$ of a and $V \subseteq \mathbb{R}^m$ of $f(a)$, with $f(U) \subseteq V$, and diffeomorphisms $\alpha : U \to U'$, $\beta : V \to V'$ onto open sets such that $\alpha(a) = 0$, $\beta(f(a)) = 0$, and

$$(\beta \circ f \circ \alpha^{-1})(u, v) = (u, 0) \quad \text{for every } (u, v) \in U' \subseteq \mathbb{R}^r \times \mathbb{R}^{n-r}.$$

(1) Translate both points to zero. If $r > 0$, permute source and target coordinates so that the matrix

$$\frac{\partial(f_1, \dots, f_r)}{\partial(x_1, \dots, x_r)}$$

is invertible at zero. Apply the inverse function theorem to

$$\alpha(x) = (f_1(x), \dots, f_r(x), x_{r+1}, \dots, x_n).$$

(2) Write $f \circ \alpha^{-1}(u, v) = (u, g(u, v))$ on a small product of open balls. Show from the rank assumption that $d_v g = 0$, and hence $g(u, v) = h(u)$.

(3) Use $\beta(y, z) = (y, z - h(y))$ to finish. Treat $r = 0$ separately by showing that f is constant on a small ball.

A slightly more general version for C^1 maps is given in [3, Theorem 9.32]. The same argument works for C^k maps with $k \geq 1$, giving C^k coordinate changes. For the formulation using linear projections in the 2025 notes, see [2, Chapter 1, "The Rank Theorem"].

1.3.7. (**Smooth retractions**) A **retraction** of an open set $U \subseteq \mathbb{R}^n$ onto $A \subseteq U$ is a map $r : U \longrightarrow A$ with $r|_A = \mathrm{Id}_A$. Viewed as a map into U, it is **idempotent**: $r \circ r = r$.

(1) If $M \subseteq \mathbb{R}^n$ is a manifold and $p \in M$, use a straightening chart, restricted so that its image is a product $A \times B$ with $0 \in B$, to construct a smooth retraction of an open neighborhood $U \subseteq \mathbb{R}^n$ of p onto $M \cap U$. Hint: conjugate $(u, v) \mapsto (u, 0)$ by the chart.

(2) Conversely, suppose $r : U \longrightarrow U$ is smooth and idempotent. Show that $r(U) = \{x \in U \mid r(x) = x\}$, and, at every fixed point p, $\mathrm{d}r(p)^2 = \mathrm{d}r(p)$. Deduce that its rank equals its trace. Prove that each fixed point p has an open neighborhood $W_0 \subseteq U$ such that $\operatorname{rank} \mathrm{d}r(q) = \operatorname{rank} \mathrm{d}r(p)$ for every fixed point $q \in W_0$.

(3) Fix $p \in r(U)$ and put $d = \operatorname{rank} \mathrm{d}r(p)$. Show that there is an open neighborhood $W \subseteq U$ of p on which $\operatorname{rank} \mathrm{d}r(x) = d$ for every $x \in W$. Hint: choose an open neighborhood $W_1 \subseteq U$ of p on which the rank is at least d, and take $W = W_1 \cap W_0 \cap r^{-1}(W_0)$. Use

$$\mathrm{d}r(r(x))\, \mathrm{d}r(x) = \mathrm{d}r(x), \qquad x \in U,$$

for the reverse inequality.

(4) Apply Exercise 1.3.6 to find an open neighborhood $V \subseteq U$ of p such that $r(U) \cap V$ is a d-dimensional manifold. Explain the necessary restriction of the target neighborhood: if O is the source neighborhood in the rank theorem, choose $V \subseteq O$ in its target neighborhood so that the coordinate slice in V is contained in $r(O)$. Every fixed point in V is its own image. Conclude that images of smooth idempotents are locally manifolds, although their dimensions may differ between components.

A global theorem, proved using the so-called *tubular neighborhoods*, says that every embedded manifold $M \subseteq \mathbb{R}^n$ is a smooth retract of some open neighborhood of M. In fact, for every open set $O \subseteq \mathbb{R}^n$ containing M, there is a tubular neighborhood V with $M \subseteq V \subseteq O$. This is a more sophisticated theorem in differential topology. Its development uses vector fields and flows, that is, ordinary differential equations formulated on manifolds; see [1, Chapter II, Sections 8 and 11, especially Theorem 11.14 and the paragraph following it]. It is not required in this course.

References

[1] Glen E. Bredon, *Topology and geometry*, Corr. 3rd printing of the 1993 original, Grad. Texts Math., vol. 139, Berlin: Springer, 1997 (English).

[2] Class of Excellence in Mathematics, HIT, *Lecture notes on mathematical analysis III*, 2025. Autumn semester course notes, available at `https://noncommutative.xyz/~hw/teaching/mathematical-analysis-iii-2026/Mathematical_Analysis_III-2025_autumn.pdf`.

[3] Walter Rudin, *Principles of mathematical analysis*, 3rd ed., International Series in Pure and Applied Mathematics, McGraw-Hill Book Company, Düsseldorf, 1976 (English).