Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD03 — Solutions

Exercise numbers agree with Chapter 1 of the lecture notes. The part numbers agree with TD03. References to propositions refer to the lecture notes. In the topological case, C^0 means continuous. For positive orders, differentiability on subsets is understood in the extension sense.

1.4 Maps and functions on manifolds and constrained extrema

1.4.1. The case $l = 0$. For Proposition 4.5, if f is continuous, choose any chart (U, φ) about a given $p \in M$. The composition $f \circ \varphi^{-1}$ is continuous. Conversely, take such a chart about each p. On its domain,

$$f|_U = (f \circ \varphi^{-1}) \circ \varphi$$

is continuous. These domains form an open cover of M, so f is continuous on M.

For Proposition 4.7, suppose first that f is continuous. Given $p \in M$, choose a chart (V, ψ) about $f(p)$ and a chart (U_0, φ_0) about p. The set $U = U_0 \cap f^{-1}(V)$ is an open neighborhood of p, so $(U, \varphi_0|_U)$ is a chart, and $f(U) \subseteq V$. Its coordinate form is a composition of continuous maps. Conversely, use the chart pair supplied at every p by condition (ii). On each such U,

$$f|_U = \psi^{-1} \circ (\psi \circ f \circ \varphi^{-1}) \circ \varphi$$

is continuous. Again these domains form an open cover, so f is continuous. This proof uses only that the charts are homeomorphisms, and therefore also applies when $k = 0$.

1.4.2. Restriction of source and target. The map in question is $f|_U : U \to V$, with its target restricted to V because $f(U) \subseteq V$. For $l \geq 1$, let F be a C^l extension of f to an open neighborhood of M in $\mathbb{R}^m$, with values in $\mathbb{R}^n$. The same F is an extension of $f|_U$, so the latter belongs to $C^l(U, V)$. It is not necessary that F take its values in V: the definition only requires its restriction to U to do so. Equivalently, one may restrict local extensions at each point of U.

For $l = 0$, let H be open in V. Write $H = V \cap G$ with G open in N. Then

$$(f|_U)^{-1}(H) = U \cap f^{-1}(G),$$

which is open in U. Thus $f|_U : U \to V$ is continuous.

1.4.3. Testing regularity using an atlas.

(1) If $f \in C^k(M, \mathbb{K})$, choose, for each $p \in M$, a chart $(U, \varphi) \in \mathcal{A}$ with $p \in U$. Composition with the chart inverse gives $f \circ \varphi^{-1} \in C^k(\varphi(U), \mathbb{K})$. Conversely, the assumed chart at each p satisfies condition (ii) of Proposition 4.5, so $f \in C^k(M, \mathbb{K})$.

(2) Suppose first that $f \in C^k(M, N)$. For each $p \in M$, choose $(U, \varphi) \in \mathcal{A}$ with $p \in U$ and $(V, \psi) \in \mathcal{B}$ with $f(p) \in V$. Since f is continuous, $W = U \cap f^{-1}(V)$ is open in M and contains p. Restriction and composition show that $\psi \circ f \circ \varphi^{-1}$ is of class C^k on $\varphi(W)$.

Conversely, choose U, V, and W as in the hypothesis at each p. The restricted chart $(W, \varphi|_W)$, together with (V, ψ), satisfies condition (ii) of Proposition 4.7. Consequently, $f \in C^k(M, N)$. Both directions also apply to $k = 0$, using continuity in place of differentiability.

When $k = \infty$, apply the cited propositions at every finite order to obtain the same conclusions.

1.4.4. Extrema of xy on an ellipse. The constraint set is nonempty and compact, so both extrema exist. For $g(x, y) = x^2 + 4y^2$, the gradient $(2x, 8y)$ is nonzero on $g^{-1}(8)$. At an extremum the multiplier equations are

$$y = 2\lambda x, \qquad x = 8\lambda y, \qquad x^2 + 4y^2 = 8.$$

Neither x nor y can vanish in this system. Substitution gives $16\lambda^2 = 1$. If $\lambda = 1/4$, then $x = 2y$, giving $(x, y) = (2, 1)$ or $(-2, -1)$, with value 2. If $\lambda = -1/4$, then $x = -2y$, giving $(x, y) = (2, -1)$ or $(-2, 1)$, with value -2. Therefore the maximum is 2, attained exactly at the first two points, and the minimum is -2, attained exactly at the last two points.

1.4.5. Extrema of xyz with two constraints. The constraint set is a nonempty compact circle. The two constraint gradients are $(1, 1, 1)$ and $2(x, y, z)$. If they were dependent, we would have $x = y = z$, and $x + y + z = 0$ would contradict $x^2 + y^2 + z^2 = 6$. Thus every extremum satisfies

$$yz = \alpha + 2\beta x, \qquad xz = \alpha + 2\beta y, \qquad xy = \alpha + 2\beta z.$$

Subtracting pairs of equations gives

$$(y - x)(z + 2\beta) = 0, \qquad (z - y)(x + 2\beta) = 0, \qquad (x - z)(y + 2\beta) = 0.$$

If the three coordinates were distinct, these identities would force all of them to equal -2β, a contradiction. At least two are equal. Write a permutation of the triple as $(a, a, -2a)$. The second constraint gives $6a^2 = 6$, so $a = \pm 1$, and the product is $-2a^3$. Comparing all candidates,

$$\max f = 2 \quad \text{at the permutations of } (2, -1, -1),$$

$$\min f = -2 \quad \text{at the permutations of } (-2, 1, 1).$$

These are all the points of attainment.

1.4.6. Diagonalization of symmetric matrices. Write $\langle u, v\rangle$ for the Euclidean inner product. Since A is symmetric,

$$\mathrm{d}f(x)h = \langle Ah, x\rangle + \langle Ax, h\rangle = 2\langle Ax, h\rangle, \qquad \operatorname{grad} f(x) = 2Ax.$$

(1) The unit sphere S is nonempty and compact, so the continuous function f attains a maximum $\lambda_1 = f(v_1)$ at some $v_1 \in S$. Since $\operatorname{grad}\Phi(v_1) = 2v_1 \neq 0$, the multiplier equations, writing $A = (a_{ij})$, are

$$2\sum_{j=1}^{n} a_{ij} v_{j,1} = 2\mu v_{i,1} \quad (1 \le i \le n), \qquad \sum_{i=1}^{n} v_{i,1}^2 = 1.$$

Thus $Av_1 = \mu v_1$. Taking the inner product with v_1 shows $\mu = f(v_1) = \lambda_1$. This proves all three assertions in part (1).

(2) By definition,

$$S_2 = \{x \in \mathbb{R}^n : \|x\|^2 = 1,\ \langle x, v_1\rangle = 0\} = \Phi_1^{-1}(1, 0).$$

It is nonempty because $v_1^{\perp}$ has dimension $n - 1 \ge 1$, and it is closed and bounded. At a point $x \in S_2$, the vectors $2x$ and v_1 are linearly independent. The regular level set theorem therefore makes S_2 a compact smooth manifold of dimension $n - 2$. Let v_2 maximize f on S_2, and put $\lambda_2 = f(v_2)$. The multiplier equations give

$$2Av_2 = 2\mu v_2 + \nu v_1.$$

Taking the inner product with v_1 and using symmetry,

$$\nu = 2\langle Av_2, v_1\rangle = 2\langle v_2, Av_1\rangle = 2\lambda_1\langle v_2, v_1\rangle = 0.$$

Taking the inner product with v_2 gives $\mu = \lambda_2$, hence $Av_2 = \lambda_2 v_2$.

(3) The hypotheses and the nesting of the sets imply that $v_1, \dots, v_k$ are orthonormal and

$$S_{k+1} = \{x : \|x\|^2 = 1,\ \langle v_i, x\rangle = 0 \text{ for } 1 \le i \le k\} = \Phi_k^{-1}(1, 0, \dots, 0).$$

At every point x of this level set,

$$\mathrm{d}\Phi_k(x)h = (2\langle x, h\rangle, \langle v_1, h\rangle, \dots, \langle v_k, h\rangle).$$

For any $(b_0, \dots, b_k) \in \mathbb{R}^{k+1}$, choosing $h = (b_0/2)x + \sum_{i=1}^k b_i v_i$ gives $\mathrm{d}\Phi_k(x)h = (b_0, \dots, b_k)$. This proves surjectivity. The level set is closed and bounded, and nonempty because the orthogonal complement of $\langle v_1, \dots, v_k\rangle$ has dimension $n - k \ge 1$. It is therefore a compact smooth manifold of dimension $n - k - 1$.

Choose a maximizer v_{k+1} and put $\lambda_{k+1} = f(v_{k+1})$. Lagrange multipliers give

$$2Av_{k+1} = 2\mu v_{k+1} + \sum_{i=1}^{k} \nu_i v_i.$$

For $1 \le i \le k$, symmetry and the induction hypotheses give

$$\nu_i = 2\langle Av_{k+1}, v_i\rangle = 2\langle v_{k+1}, Av_i\rangle = 0.$$

Taking the inner product with v_{k+1} gives $\mu = \lambda_{k+1}$, so $Av_{k+1} = \lambda_{k+1}v_{k+1}$.

(4) If $n = 1$, part (1) already gives an orthonormal eigenbasis. If $n \ge 2$, parts (1)–(3) construct n orthonormal eigenvectors, hence a basis. At each step,

$$\lambda_i = \max_{x \in S_i} f(x), \qquad S_i = S \cap \langle v_1, \dots, v_{i-1}\rangle^{\perp}.$$

Since $S_{i+1} \subseteq S_i$, their maximum values satisfy $\lambda_1 \ge \cdots \ge \lambda_n$.

(5) Let O have columns $v_1, \dots, v_n$. Orthonormality gives $O^{\mathsf{T}}O = I$, and the eigenvector identities give

$$AO = O\ \mathrm{diag}(\lambda_1, \dots, \lambda_n), \qquad O^{-1}AO = \mathrm{diag}(\lambda_1, \dots, \lambda_n).$$

(6) Apply the maximum construction to $-A$. It gives an orthonormal basis $w_1, \dots, w_n$ of eigenvectors of A, with eigenvalues $\mu_1 \le \cdots \le \mu_n$, such that

$$\mu_i = \min_{x \in S \cap \langle w_1, \dots, w_{i-1}\rangle^{\perp}} f(x).$$

Indeed, maximizing $-f$ is the same as minimizing f. One may take $w_i = v_{n+1-i}$ and $\mu_i = \lambda_{n+1-i}$: for $x = \sum_{j=1}^n c_j v_j$,

$$f(x) = \sum_{j=1}^{n} \lambda_j c_j^2, \qquad \|x\|^2 = \sum_{j=1}^{n} c_j^2,$$

which proves the stated successive minimum property.

(7) Symmetry is used in $\mathrm{grad}\, f(x) = 2Ax$, and in the identities $\langle Av_j, v_i\rangle = \langle v_j, Av_i\rangle$ that eliminate the multipliers of the orthogonality constraints. Without symmetry, the gradient is $(A + A^{\mathsf{T}})x$.

(8) Take

$$M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x_0 = \frac{1}{\sqrt{2}}(1, 1).$$

This matrix is nonsymmetric and $M^2 = 0$. For $x = (s, t)$ with $s^2 + t^2 = 1$,

$$\langle Mx, x\rangle = st = \frac{1 - (s - t)^2}{2} \le \frac{1}{2}.$$

Equality holds at x_0, but $Mx_0 = (1, 0)/\sqrt{2}$ is not a scalar multiple of x_0. Thus x_0 is not an eigenvector.