Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD03

Exercise numbers agree with Chapter 1 of the lecture notes. References to propositions and their conditions refer to the lecture notes.

1.4 Maps and functions on manifolds and constrained extrema

1.4.1. Prove the $l = 0$ case of Proposition 4.5 and Proposition 4.7, including when $k = 0$. In the latter, explain how continuity gives suitable chart pairs around every point.

1.4.2. In the proof that (i) implies (ii) in Proposition 4.7, let (U, φ) and (V, ψ) be charts about p and $f(p)$, with $f(U) \subseteq V$. Explain in detail why $f \in C^l(M, N)$ implies $f|_U \in C^l(U, V)$.

1.4.3. Let $M \subseteq \mathbb{R}^m$ be a C^k-manifold, where $k \in \mathbb{N}_\infty$. An **atlas** on M is a family of C^k-charts whose domains cover M. Let $\mathcal{A}$ be an atlas on M.

(1) Show that a function $f : M \to \mathbb{K}$ belongs to $C^k(M, \mathbb{K})$ if and only if, for every $p \in M$, there exists $(U, \varphi) \in \mathcal{A}$ with $p \in U$ such that

$$f \circ \varphi^{-1} \in C^k(\varphi(U), \mathbb{K}).$$

(2) Let $N \subseteq \mathbb{R}^n$ be a C^k-manifold with an atlas $\mathcal{B}$. Show that a map $f : M \to N$ belongs to $C^k(M, N)$ if and only if, for every $p \in M$, there exist $(U, \varphi) \in \mathcal{A}$, $(V, \psi) \in \mathcal{B}$, and an open neighborhood W of p in M, with $W \subseteq U$ and $f(W) \subseteq V$, such that the map

$$\psi \circ f \circ \varphi^{-1} : \varphi(W) \to \psi(V)$$

is of class C^k.

1.4.4. Use the method of Lagrange multipliers to find the maximum and minimum of $f(x, y) = xy$ under the constraint

$$x^2 + 4y^2 = 8.$$

Determine all points at which these values are attained.

1.4.5. Use the method of Lagrange multipliers to find the maximum and minimum of $f(x, y, z) = xyz$ under the two constraints

$$x + y + z = 0, \qquad x^2 + y^2 + z^2 = 6.$$

Determine all points at which these values are attained.

1.4.6. (**Diagonalization of symmetric matrices**) Fix an $n \in \mathbb{N}^+$. The goal of this exercise is to apply the method of Lagrange multipliers to re-derive the following fact in linear algebra: for any symmetric matrix $A \in \mathrm{Mat}_n(\mathbb{R})$, there exists an orthogonal matrix $O \in \mathrm{Mat}_n(\mathbb{R})$ such that $O^{-1}AO$ is diagonal.

(1) Let $E = \mathbb{R}^n$, equipped with the usual ℓ^2-norm, and set $S = \{x \in E \mid |x| = 1\}$. Consider the function

$$\begin{aligned} f : E &\to \mathbb{R} \\ x &\mapsto \langle Ax, x \rangle \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on E. Note that $S = \Phi^{-1}(1)$ is exactly the $(n-1)$-dimensional unit sphere, where $\Phi(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2$.

(1.a) Show that $f|_S$ achieves its maximum λ_1 at some point $v_1 \in S$.

(1.b) Let $v_1 = (v_{1,1}, v_{2,1}, \ldots, v_{n,1}) \in S$. Find a system of equations for the coordinates $(v_{i,1})_{1 \le i \le n}$ of v_1 using the method of Lagrange multipliers.

(1.c) Show that we must have $Av_1 = \lambda_1 v_1$.

(2) Assume $n \geq 2$. Set $S_1 \coloneqq S$ and $S_2 \coloneqq S_1 \cap v_1^\perp$. Define $p_1(x) \coloneqq \langle x, v_1 \rangle$, $x \in E$. Consider the map

$$\begin{aligned} \Phi_1 : E &\to \mathbb{R}^2 \\ x &\mapsto \big(\Phi(x), p_1(x)\big). \end{aligned}$$

(2.a) Show that $S_2 = \Phi_1^{-1}(1, 0)$.

(2.b) Show that S_2 is a *compact* $(n-2)$-dimensional smooth submanifold of E.

(2.c) Show that $f|_{S_2}$ achieves its maximum λ_2 at some point $v_2 \in S_2$.

(2.d) Show that $Av_2 = \lambda_2 v_2$.

(3) Suppose we have found unit vectors $v_1, \ldots, v_k$, and compact smooth submanifolds $S_1, S_2, \ldots, S_k$, as well as real numbers $\lambda_1, \ldots, \lambda_k$ ($k \in \{2, \ldots, n-1\}$), such that

$$\forall i \in \{1, \ldots, k\}, \qquad v_i \in S_i, \qquad \lambda_i = \max f(S_i) \qquad \text{and} \qquad Av_i = \lambda_i v_i$$

and

$$\forall i \in \{2, \ldots, k\}, \qquad S_i = S_{i-1} \cap v_{i-1}^\perp.$$

Here $S_1, v_1, v_2, \lambda_1, \lambda_2$ are just as before, and $S_2 \coloneqq S_1 \cap v_1^\perp$.

Define $p_i(x) \coloneqq \langle v_i, x \rangle$ for all $i \in \{1, \ldots, k\}$ and all $x \in E$, and consider the map

$$\begin{aligned} \Phi_k : E &\to \mathbb{R}^{1+k} \\ x &\mapsto (\Phi(x), p_1(x), \ldots, p_k(x)). \end{aligned}$$

Set $S_{k+1} \coloneqq S_k \cap v_k^\perp$.

(3.a) Show that $\mathrm{d}\Phi_k(x)$ is surjective for each $x \in \Phi_k^{-1}(1, 0, \ldots, 0)$.

(3.b) Show that S_{k+1} is an $(n-k-1)$-dimensional compact smooth submanifold of E.

(3.c) Show that $f|_{S_{k+1}}$ achieves its maximum λ_{k+1} at some $v_{k+1} \in S_{k+1}$.

(3.d) Show, using the method of Lagrange multipliers, that $Av_{k+1} = \lambda_{k+1} v_{k+1}$.

(4) Show that by induction, there exists an orthonormal basis $v_1, \ldots, v_n$ of E, and a non-increasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ of real numbers, such that

$$\forall i \in \{1, \ldots, n\}, \qquad \lambda_i = \max f(S \cap \langle v_1, \ldots, v_{i-1} \rangle^\perp)$$

(we make the convention that $\langle \emptyset \rangle = \{0\}$), and that $\lambda_i = f(v_i)$, as well as $Av_i = \lambda_i v_i$.

(5) Conclude by showing that taking O to be the matrix obtained by putting each v_i into the i-th column, $O^{-1}AO$ becomes the diagonal matrix whose diagonal entries are exactly $\lambda_1, \lambda_2, \ldots, \lambda_n$ consecutively.

(6) One may also repeat the above process by considering minimum values. Can you state the corresponding result and explain how one can derive the corresponding result quickly using merely the above reasoning?

Hint: replace A with $-A$.

(7) Point out where A being symmetric is used in your answer.

(8) Find an example of a non-symmetric $M \in \mathrm{Mat}_2(\mathbb{R})$ and an $x_0 \in \mathbb{R}^2$ with $|x_0| = 1$, such that we have

$$\langle Mx_0,\, x_0 \rangle = \max_{x \in \mathbb{R}^2, |x|=1} \langle Mx,\, x \rangle,$$

but x_0 is *not* an eigenvector for M.

A possible trick: by considering a nilpotent matrix, you can actually construct a very simple example almost without computation. Of course, you don't have to follow this trick.

Remark. You may now ask a very natural and obvious question: why bother using this analytic method to establish the algebraic result that all symmetric real matrices can be diagonalized? Besides its subjective elegance and the obvious excuse for practicing the method of Lagrange multipliers, one possible good answer is, while the algebraic method that you have seen in your first year's linear algebra course is of course no more difficult, it is this more analytic method of trying to maximize a suitable function on the unit sphere that generalizes easily to infinite dimensional cases, most importantly, to self-adjoint compact operators on Hilbert spaces. The corresponding infinite dimensional result plays an important role in various branches in mathematics, e.g. the theory of partial differential equations and theory of operator algebras/noncommutative geometry.