# Mathematical Analysis III, 2026 autumn semester
# HIT, class of excellence

# TD04 — Solutions

Exercise numbers agree with Chapter 2 of the lecture notes. The part and subpart numbers agree with TD04. All functions are real-valued. Before Exercise 2.1.5, integration arguments use only compactly supported continuous bounds and their integrals.

## 2.1 Iterated Integration and Fubini's Theorem

Write $I(f) = \int_{\mathbb{R}^n} f$ for the integral of a compactly supported continuous function, defined by iterated one-variable integration. Fubini's theorem for these functions makes $I$ independent of the order of integration; it is linear and positive. For any real-valued function $f$, put

$$A(f) = \sup\{I(h) : h \in C_c(\mathbb{R}^n),\ h \le f\},$$
$$B(f) = \inf\{I(g) : g \in C_c(\mathbb{R}^n),\ f \le g\}.$$

The definition in the notes says that $f$ is Riemann integrable if, for every $\varepsilon > 0$, there are $h, g \in C_c(\mathbb{R}^n)$ with $h \le f \le g$ and $I(g - h) < \varepsilon$. Then $A(f) = B(f)$, and this common value is its integral. Such an $f$ is necessarily bounded and compactly supported: any one pair $h \le f \le g$ proves both assertions.

**2.1.1. Properties of $C_c(\mathbb{R}^n)$.**

(1) Choose $R > 0$ such that $\operatorname{supp} f \subseteq [-R, R]^n$. On the compact box $K = [-R-1, R+1]^n$, the function $f$ is bounded and uniformly continuous. Outside this box it is zero, so it is bounded on all of $\mathbb{R}^n$. Given $\varepsilon > 0$, choose $0 < \delta < 1$ from uniform continuity on $K$, using the maximum norm. If $\|x - y\|_\infty < \delta$, either both points belong to $K$, or one is outside $K$. In the latter case both points are outside $[-R, R]^n$, so $f(x) = f(y) = 0$. Thus $f$ is uniformly continuous on $\mathbb{R}^n$.

(2) The function $f + \lambda g$ is continuous, and

$$\operatorname{supp}(f + \lambda g) \subseteq \operatorname{supp} f \cup \operatorname{supp} g.$$

The right-hand side is compact. Together with the zero function, this proves that $C_c(\mathbb{R}^n)$ is a real vector space.

(3) The two implications follow from pointwise identities.

(3.a) If the subspace is closed under absolute values, then

$$f \vee g = \frac{f + g + |f - g|}{2}, \qquad f \wedge g = \frac{f + g - |f - g|}{2}$$

prove closure under maximum and minimum.

(3.b) Conversely, if the subspace is closed under maximum and minimum, then $-f$ belongs to it, and $|f| = f \vee (-f)$ does also.

(4) The absolute value of a continuous function is continuous, and $\operatorname{supp}|f| = \operatorname{supp} f$. Apply part (3) to the vector space in part (2).

**2.1.2. Strict positivity.** Suppose $f \geq 0$ is nonzero, and choose $a$ with $f(a) > 0$. By continuity, choose $0 < \varepsilon < 1$ so small that $f(x) \geq f(a)/2$ whenever $|x_j - a_j| \leq \varepsilon$ for every $j$. The function $h_\varepsilon$ in the hint is continuous, supported in $[-\varepsilon, \varepsilon]$, and satisfies $0 \leq h_\varepsilon \leq 1$. Therefore the indicated product $g$ belongs to $C_c(\mathbb{R}^n)$, and $0 \leq g \leq f$. Since $h_\varepsilon(0) > 0$, its one-variable integral is positive. Fubini's theorem and positivity give

$$I(f) \geq I(g) = \frac{f(a)}{2} \left( \int_{\mathbb{R}} h_\varepsilon(t)\, \mathrm{d}t \right)^n > 0.$$

For the precise assertion of the remark, integrate only the $i$-th coordinate, keeping the other coordinates equal to those of $a$:

$$\int_{\mathbb{R}} f(a_1, \ldots, a_{i-1}, t, a_{i+1}, \ldots, a_n)\, \mathrm{d}t \geq \frac{f(a)}{2} h_\varepsilon(0)^{n-1} \int_{\mathbb{R}} h_\varepsilon(t)\, \mathrm{d}t > 0.$$

Thus the function obtained by integrating out $x_i$ cannot vanish identically unless $f = 0$.

**2.1.3. Basic properties of Riemann integrals.**

(1) Choose continuous compactly supported bounds $h_f \leq f \leq g_f$ and $h_g \leq g \leq g_g$, with gaps $e_f = g_f - h_f \geq 0$ and $e_g = g_g - h_g \geq 0$. The maximum and minimum of the lower bounds give lower bounds for $f \vee g$ and $f \wedge g$; use the upper bounds similarly. Pointwise,

$$0 \leq (g_f \vee g_g) - (h_f \vee h_g) \leq e_f + e_g,$$
$$0 \leq (g_f \wedge g_g) - (h_f \wedge h_g) \leq e_f + e_g.$$

Their gap integrals can be made arbitrarily small. Negating the bounds also proves integrability of $-f$. The zero function is integrable, so maximum and minimum give integrability of $f_+, f_-$, and $|f| = f \vee (-f)$.

For $c \geq 0$, bound $f + cg$ between $h_f + ch_g$ and $g_f + cg_g$. For $c < 0$, use $h_f + cg_g$ and $g_f + ch_g$. The integral of the gap is $I(e_f) + |c| I(e_g)$, which can again be made arbitrarily small.

(2) The zero function belongs to $\mathcal{R}_n$, and part (1) gives closure under addition and scalar multiplication. Hence $\mathcal{R}_n$ is a vector subspace.

(3) Write $J(f)$ for the integral of an integrable function. Every continuous pair of bounds satisfies $I(h_f) \leq J(f) \leq I(g_f)$. If its gap integral is less than $\varepsilon$, both endpoints differ from $J(f)$ by less than $\varepsilon$. The bounds from part (1), together with linearity of $I$, therefore give, by letting the gap integrals tend to zero,

$$J(f + cg) = J(f) + cJ(g).$$

If $f \geq 0$, zero is an admissible lower bound, so $J(f) = A(f) \geq 0$. In particular, $f \leq g$ implies $J(f) \leq J(g)$, by positivity applied to $g - f$.

(4) Since $|f| - f = 2f_- \geq 0$, linearity and positivity imply $J(f) \leq J(|f|)$. Applying the same argument to $-f$ gives the stronger inequality $|J(f)| \leq J(|f|)$.

**2.1.4. Fubini's theorem using continuous bounds.** Let $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ be Riemann integrable, with $p, q \in \mathbb{N}_+$. As in TD04, set

$$U(x) = \inf \left\{ \int_{\mathbb{R}^q} g(y)\, \mathrm{d}y \;\middle|\; g \in C_c(\mathbb{R}^q), \quad f(x,y) \leq g(y) \text{ for all } y \in \mathbb{R}^q \right\},$$
$$L(x) = \sup \left\{ \int_{\mathbb{R}^q} h(y)\, \mathrm{d}y \;\middle|\; h \in C_c(\mathbb{R}^q), \quad h(y) \leq f(x,y) \text{ for all } y \in \mathbb{R}^q \right\}.$$

(1) Choose $h, g \in C_c(\mathbb{R}^{p+q})$ with $h \le f \le g$. For each fixed $x$, the functions $h(x,\cdot)$ and $g(x,\cdot)$ belong to $C_c(\mathbb{R}^q)$, so both defining sets are nonempty. If $a \le f(x,\cdot) \le b$ are any admissible continuous bounds, positivity gives $\int a \le \int b$. Taking the supremum and infimum, and using the chosen bounds, yields

$$\int_{\mathbb{R}^q} h(x,y)\,\mathrm{d}y \le L(x) \le U(x) \le \int_{\mathbb{R}^q} g(x,y)\,\mathrm{d}y.$$

Both endpoints are finite, proving all assertions.

(2) Given $\varepsilon > 0$, choose the bounds in part (1) with $I(g-h) < \varepsilon$. Repeatedly applying the proposition on integrating out one variable shows that

$$H(x) = \int_{\mathbb{R}^q} h(x,y)\,\mathrm{d}y, \qquad G(x) = \int_{\mathbb{R}^q} g(x,y)\,\mathrm{d}y$$

belong to $C_c(\mathbb{R}^p)$. By part (1), $H \le L \le U \le G$. Fubini's theorem for compactly supported continuous functions gives

$$I(G-H) = I(g-h) < \varepsilon.$$

These continuous bounds prove that both $L$ and $U$ are Riemann integrable by the definition in the notes.

(3) With the bounds in part (2), positivity and the definition of the integral show that $\int L$ and $\int U$ lie between $I(H) = I(h)$ and $I(G) = I(g)$. The number $\int f$ lies between these same endpoints. Their difference is less than $\varepsilon$; since $\varepsilon > 0$ is arbitrary,

$$\int_{\mathbb{R}^p} L(x)\,\mathrm{d}x = \int_{\mathbb{R}^p \times \mathbb{R}^q} f = \int_{\mathbb{R}^p} U(x)\,\mathrm{d}x.$$

(4) If $L \le F \le U$, the same bounds satisfy $H \le F \le G$ with arbitrarily small integral gap. Hence $F$ is Riemann integrable. Its integral and $\int f$ lie between $I(h)$ and $I(g)$, so the argument of part (3) gives $\int F = \int f$.

(5) If every function $y \mapsto f(x,y)$ is Riemann integrable, its integral, by definition, is the common value $L(x) = U(x)$. Taking $F(x) = \int_{\mathbb{R}^q} f(x,y)\,\mathrm{d}y$ in part (4) gives

$$\int_{\mathbb{R}^p} \left( \int_{\mathbb{R}^q} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^p \times \mathbb{R}^q} f.$$

(6) Define $\widetilde{L}(y)$ and $\widetilde{U}(y)$ by taking the supremum and infimum of integrals of continuous compactly supported bounds for $x \mapsto f(x,y)$ on $\mathbb{R}^p$. The preceding argument, with the variables interchanged, proves that they are finite, $\widetilde{L} \le \widetilde{U}$, and both are Riemann integrable on $\mathbb{R}^q$, with

$$\int_{\mathbb{R}^q} \widetilde{L}(y)\,\mathrm{d}y = \int_{\mathbb{R}^p \times \mathbb{R}^q} f = \int_{\mathbb{R}^q} \widetilde{U}(y)\,\mathrm{d}y.$$

Every function between them is integrable with the same integral. If every function $x \mapsto f(x,y)$ is Riemann integrable, the ordinary iterated integral in this order exists and equals $\int f$. Thus the two orders agree whenever both are defined.

(7) Consider the function in the hint,

$$f(x,y) = \begin{cases} 1, & x = 0 \text{ and } y \in \mathbb{Q} \cap [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Choose $\beta \in C_c(\mathbb{R})$ with $\beta \ge 0$ and $\beta = 1$ on $[0,1]$. For $\delta > 0$, the compactly supported continuous bounds

$$0 \le f(x,y) \le g_\delta(x,y) = \max\left(1 - \frac{|x|}{\delta}, 0\right) \beta(y)$$

have integral gap $I(g_\delta) = \delta \int \beta$, tending to zero. Thus $f$ is Riemann integrable on $\mathbb{R}^2$, with integral zero.

Now set $k(y) = f(0, y)$. If $a, b \in C_c(\mathbb{R})$ satisfy $a \le k \le b$, density of the rational and irrational numbers, together with continuity, implies $b \ge 1$ and $a \le 0$ throughout $[0, 1]$. Moreover $b - a \ge 0$ on all of $\mathbb{R}$. Consequently,

$$\int_{\mathbb{R}} (b - a) \ge \int_0^1 (b - a) \ge 1.$$

The gap cannot be made arbitrarily small, so $k$ is not Riemann integrable in the definition using continuous bounds. Hence the ordinary iterated integral with $y$ integrated first is undefined. This proves the counterexample without using Riemann sums.

**2.1.5. Riemann integrals via Riemann sums.** Write $v_J = \mathrm{vol}(P_J)$, $M_J = \sup f(P_J)$, and $m_J = \inf f(P_J)$. Put $M = \|f\|_\infty$ and $V = (2R)^n$. Multiplying the one-dimensional length identities gives

$$\sum_J v_J = \prod_{i=1}^{n} \left( \sum_{j=0}^{m_i - 1} (a_{i,j+1} - a_{i,j}) \right) = V.$$

For continuous functions on a box, integrals over that box are understood as iterated one-variable integrals. Their positivity and additivity over a rectangular partition follow from the corresponding one-variable properties.

(1) **Partitions and rectangles.**

(1.a) Set $P_i^{(3)} = P_i^{(1)} \cup P_i^{(2)}$. Each is a finite partition with the same endpoints, so their product is a common refinement.

(1.b) Distinct cells differ in at least one coordinate interval. In that coordinate their intervals have disjoint interiors; if they meet, they meet at a common endpoint. Every point in the cell intersection therefore has a boundary coordinate for both cells, proving the required inclusion.

(2) **Upper and lower Riemann sums.**

(2.a) The definition is equivalently

$$\overline{S}(f, P) = \sum_J M_J v_J.$$

All suprema are finite because $f$ is bounded, and every cell has the stated product of side lengths as its volume.

(2.b) If $Q$ refines $P$, each cell of $P$ is subdivided into cells of $Q$ whose volumes add to its volume. On every smaller cell the supremum of $f$ is at most the supremum on its containing cell. Multiply by the smaller cell volumes and sum, first within each $P$-cell and then over all $P$-cells. This gives $\overline{S}(f, Q) \le \overline{S}(f, P)$.

(2.c) The lower sum is $\underline{S}(f, P) = \sum_J m_J v_J$. Since $\inf_{P_J} f = -\sup_{P_J}(-f)$, summation gives $\underline{S}(f, P) = -\overline{S}(-f, P)$.

(2.d) Apply part (2.b) to $-f$ and use part (2.c):

$$\underline{S}(f, Q) = -\overline{S}(-f, Q) \ge -\overline{S}(-f, P) = \underline{S}(f, P).$$

(2.e) Since $-M \le m_J \le M_J \le M$, multiplication by $v_J$ and summation give

$$-MV \le \underline{S}(f, P) \le \overline{S}(f, P) \le MV.$$

(2.f) Let $h \in C_c(\mathbb{R}^n)$ with $f \le h$, and fix $\varepsilon > 0$. Uniform continuity gives $\delta > 0$ such that the oscillation of $h$ on a cell of maximum-norm diameter less than $\delta$ is at most $\varepsilon/V$. Choose a partition $P$ of $\square_R$ with all coordinate intervals shorter than $\delta$. Integrating the continuous function $h$ on each cell gives

$$\overline{S}(f,P) \le \overline{S}(h,P) \le \int_{\square_R} h + \varepsilon \le \int_{\mathbb{R}^n} h + \varepsilon.$$

The middle inequality follows by bounding the difference between the supremum and the values of $h$ on each cell by $\varepsilon/V$. The last follows because $h \ge f = 0$ outside $\square_R$; its integral can be computed on a larger bounded box containing its support.

(3) **Upper and lower Riemann integrals.**

(3.a) The upper integral is the infimum of the upper sums, hence is finite by part (2.e). For every continuous compactly supported majorant $g$, part (2.f) gives partitions with upper sum at most $I(g) + \varepsilon$, for arbitrary $\varepsilon > 0$. Thus $\overline{\int} f \le I(g)$. Taking the infimum over majorants proves $\overline{\int} f \le B(f)$.

(3.b) Part (2.c) implies $\underline{\int} f = -\overline{\int}(-f)$. Apply part (3.a) to $-f$. Negation gives a bijection between its continuous majorants and the continuous minorants of $f$, so $\underline{\int} f \ge A(f)$.

(3.c) Given $P, Q$, choose a common refinement $T$ by part (1.a). Parts (2.b), (2.d), and (2.e) give

$$\underline{S}(f,P) \le \underline{S}(f,T) \le \overline{S}(f,T) \le \overline{S}(f,Q).$$

Taking the supremum over $P$ and the infimum over $Q$ proves $\underline{\int} f \le \overline{\int} f$.

(3.d) Consider $0 < r < s$ with $\operatorname{supp} f \subseteq \square_r$. Write $U_r, L_r$ and $U_s, L_s$ for the integrals computed using the respective boxes. Refine a partition of $\square_s$ to include the faces of $\square_r$, and restrict it to $\square_r$. Every exterior cell contains points outside $\square_r$, where $f = 0$, so its supremum is nonnegative and its infimum nonpositive. Refinement therefore shows that every upper sum on $\square_s$ is at least $U_r$, and every lower sum is at most $L_r$. Hence $U_s \ge U_r$ and $L_s \le L_r$.

Conversely, extend any partition of $\square_r$ to $\square_s$, adding the coordinate points $\pm(r+\delta)$, where $0 < \delta < s - r$. Exterior cells on which $f$ is not identically zero must touch $\square_r$, so all such cells lie in the collar between $\square_r$ and $\square_{r+\delta}$. Their total volume is at most $(2(r+\delta))^n - (2r)^n$. Since $|f| \le M$, the added upper-sum contribution is between zero and $M((2(r+\delta))^n - (2r)^n)$, and the added lower-sum contribution is between its negative and zero. This quantity tends to zero with $\delta$. Taking infima or suprema of the inner sums, then letting $\delta$ tend to zero, gives $U_s \le U_r$ and $L_s \ge L_r$. This proves independence of the box, including when $f$ does not vanish on its boundary.

(3.e) For $g \in C_c(\mathbb{R}^n)$, choose a box containing $\operatorname{supp} g$, as permitted by part (3.d). For every partition of this box, positivity on each cell gives

$$\underline{S}(g,P) \le I(g) \le \overline{S}(g,P).$$

Uniform continuity shows that, on sufficiently fine partitions, the difference of these two sums is at most $V\,\omega_g(\delta)$, where

$$\omega_g(\delta) = \sup\{|g(x) - g(y)| : x, y \in \square_R,\ \|x - y\|_\infty \le \delta\} \longrightarrow 0.$$

Thus both Darboux integrals equal $I(g)$, as claimed.

(4) **Tools for constructing continuous bounds.**

(4.a) For $0 < t < 1/2$, take

$$\widetilde{c_i} = c_i + t(d_i - c_i), \qquad \widetilde{d_i} = d_i - t(d_i - c_i).$$

The inner rectangle is nondegenerate, lies in the interior of R, and has volume $(1-2t)^n \operatorname{vol}(\mathrm{R})$. Let $t$ tend to zero to obtain the desired volume estimate.

(4.b) The sets $\operatorname{Int}(D)$ and $\mathbb{R}^n \setminus C$ form an open cover of $\mathbb{R}^n$, because $C \subseteq \operatorname{Int}(D)$. Take a subordinate partition of unity $(\psi, \chi)$ on $\mathbb{R}^n$. On $C$, subordination gives $\chi = 0$, so $\psi = 1$. On $\partial D$, it gives $\psi = 0$. Since $0 \le \psi \le 1$, the continuous function

$$\varphi(x) = c\,\psi(x) + d\,(1 - \psi(x)), \qquad x \in D,$$

has the required values, and is between $\min(c, d)$ and $\max(c, d)$. This also covers empty sets and $c = d$.

(4.c) Compatibility makes $\varphi(x)$ independent of the index chosen. If $D \subseteq Y$ is closed, then $\varphi_i^{-1}(D)$ is closed in $C_i$, hence in $X$, since $C_i$ is closed. The finite union

$$\varphi^{-1}(D) = \bigcup_{i=1}^{m} \varphi_i^{-1}(D)$$

is closed in $X$. Thus $\varphi$ is continuous.

(5) **Construction of a continuous majorant.**

(5.a) Part (4.a) gives inner rectangles $Q_J$ with arbitrarily small volume loss. Apply part (4.b) to $C = Q_J$, $D = P_J$, $c = M_J$, and $d = M$. Since $M_J \le M$, the resulting continuous function $g_J$ satisfies

$$g_J = M_J \text{ on } Q_J, \qquad g_J = M \text{ on } \partial P_J, \qquad M_J \le g_J \le M.$$

(5.b) Part (1.b) shows that cell intersections lie in both boundaries, where the functions all equal $M$. Part (4.c) therefore pastes them into a continuous $g_0$ on $\square_R$. On each cell, $g_0 \ge M_J \ge f$ and $-M \le g_0 \le M$. Every point of $\partial\square_R$ is on a cell boundary, so $g_0 = M$ there.

(5.c) On $P_J$, the continuous function $g_J - M_J$ lies between zero and $2M$, and vanishes on $Q_J$. Subdivide $P_J$ along the faces of $Q_J$. Positivity and additivity of the continuous integrals give

$$0 \le \int_{P_J} g_J - M_J v_J \le 2M\big(v_J - \operatorname{vol}(Q_J)\big).$$

Summing over all cells proves the stated estimate for $g_0$.

(5.d) Apply part (4.b) with $C = \square_R$, the chosen larger box $D$, $c = M$, and $d = 0$. It gives a continuous $\varphi$ on $D$ with $\varphi = M$ on $C$, $\varphi = 0$ on $\partial D$, and $0 \le \varphi \le M$. Define $g$ by $g_0$ on $C$, by $\varphi$ on $D \setminus \operatorname{Int}(C)$, and by zero on $\mathbb{R}^n \setminus \operatorname{Int}(D)$. These three closed sets cover $\mathbb{R}^n$; their definitions agree on their intersections by the boundary values. Part (4.c) proves continuity. Since $g = 0$ outside the compact box $D$, it belongs to $C_c(\mathbb{R}^n)$. On the exterior of $C$, it is between zero and $M$. Subdivide $D$ along the faces of $C$ and integrate to get

$$0 \le I(g) - \int_{\square_R} g_0 \le M\big(\operatorname{vol}(D) - V\big).$$

(5.e) If $M = 0$, then $f = 0$, and take $g = 0$. Otherwise, let $N$ be the number of cells. Choose the inner rectangles and the larger box so that

$$v_J - \operatorname{vol}(Q_J) < \frac{\varepsilon}{4MN}, \qquad M\big(\operatorname{vol}(D) - V\big) < \frac{\varepsilon}{2}.$$

The first choices are possible by part (4.a); for the second, take $D = \square_{R+\eta}$ with $\eta > 0$ small. Combining parts (5.c) and (5.d) gives

$$I(g) - \overline{S}(f, P) \leq 2M \sum_J \big(v_J - \mathrm{vol}(Q_J)\big) + M\big(\mathrm{vol}(D) - V\big) < \varepsilon.$$

The boundary values are those of $g_0$. On $\square_R$ we have $g = g_0 \geq f$, and outside it $g \geq 0 = f$. Thus all required properties hold.

(6) **Equivalence of the two definitions and boundedness.**

(6.a) Part (3.a) gives $\overline{\int} f \leq B(f)$. Given $\varepsilon > 0$, choose $P$ with $\overline{S}(f, P) < \overline{\int} f + \varepsilon/2$. Apply part (5.e) with $\varepsilon/2$ to obtain a continuous compactly supported majorant $g$ such that

$$B(f) \leq I(g) \leq \overline{S}(f, P) + \varepsilon/2 < \overline{\int} f + \varepsilon.$$

Let $\varepsilon$ tend to zero to obtain equality. Apply it to $-f$, using part (2.c), to get $\underline{\int} f = A(f)$.

(6.b) If $f$ is integrable by continuous bounds, part (6.a) gives $\underline{\int} f = A(f) = B(f) = \overline{\int} f$. Conversely, if the two Darboux integrals have common value $a$, part (6.a) supplies continuous compactly supported bounds $h \leq f \leq g$ with

$$I(h) > a - \varepsilon/2, \qquad I(g) < a + \varepsilon/2.$$

Hence $I(g - h) < \varepsilon$. This proves integrability by continuous bounds, and its integral is the common value $a$.

(6.c) If $f$ is unbounded above, some cell of every finite partition has supremum $+\infty$, since the cells cover $\square_R$, outside which $f = 0$. Its volume is positive, so the upper sum is $+\infty$. No cell has supremum $-\infty$, because $f$ is real-valued and cells are nonempty. Thus this sum is well-defined even if $f$ is also unbounded below. Negating the function proves the corresponding assertion for lower sums. In particular, unboundedness prevents the upper and lower integrals from coinciding at a finite value, whereas compact support and boundedness permit the finite estimates used above.