Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD04

Exercise numbers agree with Chapter 2 of the lecture notes. References to definitions, propositions, remarks, and theorems refer to the lecture notes.

2.1 Iterated Integration and Fubini's Theorem

All functions are real-valued unless specified otherwise. Write $C_c(\mathbb{R}^n)$ for the space of compactly supported continuous functions on $\mathbb{R}^n$.

2.1.1. Show the following properties of $C_c(\mathbb{R}^n)$.

(1) Show that any function in $C_c(\mathbb{R}^n)$ is bounded and uniformly continuous.

(2) If $f, g \in C_c(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, then $f + \lambda g \in C_c(\mathbb{R}^n)$, i.e. $C_c(\mathbb{R}^n)$ is a real vector space.

(3) Suppose $f, g : X \to \mathbb{R}$ are functions on X. Define $f \vee g, f \wedge g : X \to \mathbb{R}$ by

$$(f \vee g)(x) = \max(f(x), g(x)) \qquad \text{and} \qquad (f \wedge g)(x) = \min(f(x), g(x)).$$

Let $\mathfrak{F}(X)$ be a vector subspace of the space of all functions on X. Show that the following are equivalent:

(3.a) For each $f \in \mathfrak{F}(X)$, we have $|f| \in \mathfrak{F}(X)$.

(3.b) For all $f, g \in \mathfrak{F}(X)$, we have $f \vee g, f \wedge g \in \mathfrak{F}(X)$. (This means that $\mathfrak{F}(X)$ is a vector lattice in mathematical jargon.)

Hint: note that for all $a, b \in \mathbb{R}$, we have

$$2\max(a, b) = a + b + |a - b| \qquad \text{and} \qquad \max(a, b) + \min(a, b) = a + b.$$

(4) Show that for any $f, g \in C_c(\mathbb{R}^n)$, we have $f \vee g, f \wedge g \in C_c(\mathbb{R}^n)$.

2.1.2. Prove Remark 1.2.

Hint: One possible approach goes as follows: for each *non-zero* $f \in C_c(\mathbb{R}^n)$ with $f \geq 0$, if $f(a) > 0$, show that there exists a small $\varepsilon > 0$, such that the function

$$g(x) \coloneqq \frac{1}{2} f(a) h_\varepsilon(x_1 - a_1) h_\varepsilon(x_2 - a_2) \cdots h_\varepsilon(x_n - a_n)$$

satisfies $f \geq g$, where $h_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$h_\varepsilon(t) = \begin{cases} \exp\left(-\dfrac{1}{\varepsilon^2 - t^2}\right), & t \in \left]-\varepsilon, \varepsilon\right[; \\ 0, & \text{otherwise.} \end{cases}$$

Then show $\int f \geq \int g > 0$.

2.1.3. (**Basic properties of Riemann integrals of Riemann integrable functions**) Recall Definition 1.5.

(1) Show that if $f, g : \mathbb{R}^n \to \mathbb{R}$ are Riemann integrable (Definition 1.5), then the functions

$$f \vee g,\ f \wedge g,\ f_+ \coloneqq f \vee 0,\ f_- \coloneqq (-f) \vee 0 = -(f \wedge 0),\ |f| \text{ and } f + cg$$

are all Riemann integrable, where $c \in \mathbb{R}$ is an arbitrary constant.

(2) Let $\mathcal{R}_n$ denote the space of all Riemann integrable functions on $\mathbb{R}^n$. Show that $\mathcal{R}_n$ is a vector subspace of the vector space of all functions on $\mathbb{R}^n$.

(3) Show that the Riemann integral

$$\int_{\mathbb{R}^n} : \mathcal{R}_n \to \mathbb{R}, \qquad f \mapsto \int_{\mathbb{R}^n} f$$

is linear and positive.

Hint: use Proposition 1.1 and Definition 1.5.

(4) Show that for any $f \in \mathcal{R}_n$, we have $\int f \leq \int |f|$.

Hint: $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

2.1.4. Let $p, q \in \mathbb{N}_+$, and let $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ be Riemann integrable in the sense of Definition 1.5. For each $x \in \mathbb{R}^p$, define

$$U(x) = \inf \left\{ \int_{\mathbb{R}^q} g(y)\,\mathrm{d}y \;\middle|\; g \in C_c(\mathbb{R}^q), \quad f(x,y) \leq g(y) \text{ for all } y \in \mathbb{R}^q \right\},$$
$$L(x) = \sup \left\{ \int_{\mathbb{R}^q} h(y)\,\mathrm{d}y \;\middle|\; h \in C_c(\mathbb{R}^q), \quad h(y) \leq f(x,y) \text{ for all } y \in \mathbb{R}^q \right\}.$$

(1) Show that $U(x)$ and $L(x)$ are finite for every $x \in \mathbb{R}^p$, and that $L \leq U$.

(2) Show that L and U are Riemann integrable on $\mathbb{R}^p$.

Hint: choose compactly supported continuous bounds for f whose integrals differ by less than an arbitrary $\varepsilon > 0$. Integrate these bounds in the y variables, and use Theorem 1.3 for compactly supported continuous functions.

(3) Show that

$$\int_{\mathbb{R}^p} L(x)\,\mathrm{d}x = \int_{\mathbb{R}^p \times \mathbb{R}^q} f = \int_{\mathbb{R}^p} U(x)\,\mathrm{d}x.$$

(4) Show that every function $F : \mathbb{R}^p \to \mathbb{R}$ with $L \leq F \leq U$ is Riemann integrable and has integral $\int_{\mathbb{R}^p \times \mathbb{R}^q} f$.

(5) Conclude that, if $y \mapsto f(x,y)$ is Riemann integrable for every $x \in \mathbb{R}^p$, then

$$\int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^q} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_{\mathbb{R}^p \times \mathbb{R}^q} f.$$

(6) State the corresponding results with the two groups of variables interchanged.

(7) Give a counterexample to the naive formulation that, for every Riemann integrable $f : \mathbb{R}^2 \to \mathbb{R}$, all functions $y \mapsto f(x,y)$, $x \in \mathbb{R}$, are Riemann integrable, and that both ordinary iterated Riemann integrals exist and equal $\int_{\mathbb{R}^2} f$.

Hint: consider

$$f(x,y) = \begin{cases} 1, & x = 0 \text{ and } y \in \mathbb{Q} \cap [0,1]; \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is Riemann integrable on $\mathbb{R}^2$, and explain why the function $y \mapsto f(0,y)$ is not Riemann integrable on $\mathbb{R}$. What does this imply for the iterated integral in which y is integrated first?

2.1.5. (Riemann Integrals via Riemann sums). Let f be a bounded function on $\mathbb{R}^n$, such that f vanishes outside a compact set, i.e. f is compactly supported. Consider a big box $\square_R \coloneqq [-R,R]^n$, $R > 0$ such that $\square_R \supseteq \operatorname{supp} f$. By a partition of $\square_R$, we mean a set $P \subseteq \mathbb{R}^n$ of the form $P = P_1 \times \cdots \times P_n$, where each P_i is a partition of $[-R,R]$, i.e. there exists $m_i \in \mathbb{N}_+$, such that

$$P_i = \left\{ a_{i,j} \;\middle|\; j = 0,1,\ldots,m_i, \text{ and } -R = a_{i,0} < a_{i,1} < \cdots < a_{i,m_i} = R \right\}.$$

We say that another partition $Q = Q_1 \times \cdots \times Q_n$ of $\square_R$ is **finer** than the partition P if $Q \supseteq P$, i.e. for each $i = 1,\ldots,n$, Q_i is finer than P_i as partitions of the interval $[-R,R]$.

(1) **Partitions and rectangles.**

(1.a) Show that for any given two partitions $P^{(s)} = P_1^{(s)} \times \cdots \times P_n^{(s)}$ of $\square_R$, $s = 1,2$, there exists a third partition $P^{(3)} = P_1^{(3)} \times \cdots \times P_n^{(3)}$ of $\square_R$ such that $P^{(3)}$ is finer than both $P^{(1)}$ and $P^{(2)}$.

(1.b) Given a partition P as above, we define the P-rectangles to be sets of the form

$$P(j_1,\ldots,j_n) = [a_{1,j_1}, a_{1,j_1+1}] \times \cdots \times [a_{n,j_n}, a_{n,j_n+1}],$$

where $0 \le j_i \le m_i - 1$. Show that given any two distinct P-rectangles $P(j_1,\ldots,j_n)$ and $P(k_1,\ldots,k_n)$, we have

$$P(j_1,\ldots,j_n) \cap P(k_1,\ldots,k_n) \subseteq \partial P(j_1,\ldots,j_n) \cap \partial P(k_1,\ldots,k_n),$$

where ∂A denotes the boundary of $A \subseteq \mathbb{R}^n$ in $\mathbb{R}^n$.

(2) **Upper and lower Riemann sums.**

(2.a) We define the **upper Riemann sum** $\overline{S}(f,P)$ of f with respect to P by

$$\overline{S}(f,P) = \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} \left[\sup f\Big(P(j_1,\ldots,j_n)\Big) \right] (a_{1,j_1+1} - a_{1,j_1}) \cdots (a_{n,j_n+1} - a_{n,j_n}).$$

(2.b) Show that if Q is finer than P, then

$$\overline{S}(f,Q) \le \overline{S}(f,P).$$

(2.c) Similarly, define the **lower Riemann sum** $\underline{S}(f,P)$ of f with respect to P by

$$\underline{S}(f,P) = \sum_{j_1=0}^{m_1-1} \cdots \sum_{j_n=0}^{m_n-1} \left[\inf f\Big(P(j_1,\ldots,j_n)\Big) \right] (a_{1,j_1+1} - a_{1,j_1}) \cdots (a_{n,j_n+1} - a_{n,j_n}).$$

Show that $\underline{S}(f,P) = -\overline{S}(-f,P)$.

(2.d) Show that if Q is finer than P, then

$$\underline{S}(f,Q) \ge \underline{S}(f,P).$$

(2.e) Show that

$$-\|f\|_\infty (2R)^n \le \underline{S}(f,P) \le \overline{S}(f,P) \le \|f\|_\infty (2R)^n.$$

(2.f) Show that if $h \in C_c(\mathbb{R}^n)$ and $f \le h$, then for every $\varepsilon > 0$, there exists a partition P of $\square_R$ such that

$$\overline{S}(f,P) \le \int_{\mathbb{R}^n} h + \varepsilon.$$

Hint: h is uniformly continuous and is supported in a larger, but still bounded box. Consider sufficiently fine partitions of that box which include the faces of $\square_R$, and note that $h \ge 0$ outside $\square_R$.

(3) **Upper and lower Riemann integrals.**

(3.a) Define the **upper Riemann integral** $\overline{\int}_{\mathbb{R}^n} f$ of f to be $\inf\{\overline{S}(f,P)\}$, where P runs through all partitions of $\square_R$. Show that

$$\overline{\int_{\mathbb{R}^n}} f \le \inf\left\{ \int_{\mathbb{R}^n} g \;\middle|\; g \in C_c(\mathbb{R}^n) \text{ and } f \le g \right\}.$$

(3.b) Define the **lower Riemann integral** $\underline{\int}_{\mathbb{R}^n} f$ of f to be $\sup\{\underline{S}(f,P)\}$ where P runs through all partitions of $\square_R$. Show that

$$\underline{\int_{\mathbb{R}^n}} f \ge \sup\left\{ \int_{\mathbb{R}^n} h \;\middle|\; h \in C_c(\mathbb{R}^n) \text{ and } h \le f \right\}.$$

(3.c) Use a common refinement to show that, for any two partitions P and Q,

$$\underline{S}(f,P) \le \overline{S}(f,Q).$$

Conclude that $\underline{\int}_{\mathbb{R}^n} f \le \overline{\int}_{\mathbb{R}^n} f$.

(3.d) Show that the upper and lower Riemann integrals of f do not depend on the choice of $R > 0$, provided that $\square_R \supseteq \operatorname{supp} f$.

(3.e) Show that if $g \in C_c(\mathbb{R}^n)$, then $\overline{\int}_{\mathbb{R}^n} g = \int_{\mathbb{R}^n} g = \underline{\int}_{\mathbb{R}^n} g$.
Hint: consider a large box containing the support of g.

(4) **Tools for constructing continuous bounds.**

(4.a) Consider any rectangle $\mathsf{R} := [c_1, d_1] \times \cdots \times [c_n, d_n]$ contained in $\mathbb{R}^n$, and define its **volume** $\operatorname{vol}(\mathsf{R})$ to be $(d_1 - c_1) \cdots (d_n - c_n)$. Suppose R is *non-degenerate* in the sense that $d_i > c_i$ for each $i = 1, \ldots, n$. Show that for any $\varepsilon > 0$, there exists a smaller non-degenerate rectangle $\widetilde{\mathsf{R}} := [\widetilde{c}_1, \widetilde{d}_1] \times \cdots \times [\widetilde{c}_n, \widetilde{d}_n]$ i.e. $\widetilde{\mathsf{R}} \subseteq \mathsf{R}$, such that

$$\operatorname{vol}(\mathsf{R}) - \operatorname{vol}(\widetilde{\mathsf{R}}) < \varepsilon.$$

(4.b) Let C, D be closed subsets of $\mathbb{R}^n$ such that $C \subseteq \operatorname{Int}(D)$. Show that for any $c, d \in \mathbb{R}$, there exists a continuous function φ on D, such that $\varphi(x) = c$ for all $x \in C$, $\varphi(x) = d$ for all $x \in \partial D$, and $\min(c,d) \le \varphi \le \max(c,d)$.
Hint: take a partition of unity (ψ, χ) on $\mathbb{R}^n$ subordinate to the open cover $\{\operatorname{Int}(D),\ \mathbb{R}^n \setminus C\}$, with $\operatorname{supp}\psi \subseteq \operatorname{Int}(D)$. Consider $c + (1-\psi)(d-c)$ on D.

(4.c) Let X, Y be topological spaces, and let $C_1, \ldots, C_m$ be *finitely many* closed subsets of X such that $X = \cup_{i=1}^m C_i$. Suppose there exists a continuous map $\varphi_i : C_i \to Y$ such that $\varphi_i|_{C_i \cap C_j} = \varphi_j|_{C_i \cap C_j}$ for all $i, j = 1, \ldots, m$. Show that the map

$$\varphi : X \to Y, \qquad x \mapsto \varphi_i(x), \text{ if } x \in C_i,$$

is well-defined and continuous.
Hint: for continuity, take a closed subset D of Y, and note that

$$\varphi^{-1}(D) = \cup_{i=1}^m \varphi_i^{-1}(D).$$

(5) **Construction of a continuous majorant.**

(5.a) For a partition P of $\square_R$, write $P_J = P(j_1, \ldots, j_n)$, where $J = (j_1, \ldots, j_n)$ and $0 \le j_i \le m_i - 1$. Put $M = \|f\|_\infty$ and $M_J = \sup f(P_J)$. Choose non-degenerate rectangles $Q_J \subseteq \mathrm{Int}(P_J)$ whose volumes are arbitrarily close to those of P_J. Show that there are continuous functions $g_J : P_J \to \mathbb{R}$ such that

$$g_J\big|_{Q_J} = M_J, \qquad g_J\big|_{\partial P_J} = M, \qquad M_J \le g_J \le M.$$

Hint: use (4.a) and (4.b).

(5.b) Show that the functions g_J agree on intersections of cells, and paste them into a continuous function $g_0 : \square_R \to \mathbb{R}$. Show that

$$f \le g_0, \qquad -M \le g_0 \le M, \qquad g_0|_{\partial \square_R} = M.$$

Hint: use (4.c).

(5.c) Show that

$$0 \le \int_{\square_R} g_0 - \overline{S}(f, P) \le 2M \sum_J \big(\mathrm{vol}(P_J) - \mathrm{vol}(Q_J)\big).$$

Hint: $g_J - M_J$ vanishes on Q_J and lies between zero and $2M$ on P_J. Subdivide along the faces of Q_J when estimating its integral.

(5.d) Let D be a slightly larger closed box whose interior contains $C = \square_R$. Show that g_0 has a continuous extension $g : \mathbb{R}^n \to \mathbb{R}$ which vanishes outside D, satisfies $0 \le g \le M$ outside C, and hence satisfies

$$0 \le \int_{\mathbb{R}^n} g - \int_{\square_R} g_0 \le M\big(\mathrm{vol}(D) - \mathrm{vol}(\square_R)\big).$$

Hint: use (4.b) with $c = M$ and $d = 0$. Keep g_0 on C, use the resulting function on $D \setminus \mathrm{Int}(C)$, and assign zero on the closed set $\mathbb{R}^n \setminus \mathrm{Int}(D)$. Apply (4.c) to these three closed sets.

(5.e) Conclude that for every $\varepsilon > 0$, there exists $g \in C_c(\mathbb{R}^n)$ such that

$$g|_{\partial \square_R} = \|f\|_\infty, \qquad f \le g, \qquad \int_{\mathbb{R}^n} g - \overline{S}(f, P) \le \varepsilon.$$

Hint: choose the inner rectangles Q_J and the outer box D sufficiently close to P_J and $\square_R$, respectively.

(6) **Equivalence of the two definitions and boundedness.**

(6.a) Show that

$$\overline{\int_{\mathbb{R}^n}} f = \inf\left\{ \int_{\mathbb{R}^n} g \;\middle|\; g \in C_c(\mathbb{R}^n) \text{ and } f \le g \right\},$$

and

$$\underline{\int_{\mathbb{R}^n}} f = \sup\left\{ \int_{\mathbb{R}^n} h \;\middle|\; h \in C_c(\mathbb{R}^n) \text{ and } h \le f \right\}.$$

Hint: the second equality follows from the first by considering (2.c).

(6.b) Conclude that f is Riemann integrable in the sense of Definition 1.5 if and only if $\overline{\int}_{\mathbb{R}^n} f = \underline{\int}_{\mathbb{R}^n} f$, in which case, this common value is exactly $\int_{\mathbb{R}^n} f$.

(6.c) Now drop the assumption of boundedness on f. Show that if f is not bounded above (resp. below), then each $\overline{S}(f,P) = +\infty$ (resp. $\underline{S}(f,P) = -\infty$).

Remark: this exercise shows why boundedness of f is necessary for the upper and lower Riemann integrals of f, defined in the same way as above, to coincide. The requirement that $\operatorname{supp} f$ be compact is an inherent assumption for the (usual) Riemann integrals. Following the Riemann integrals approach, we can develop a theory of improper Riemann integrals of several variables, when $\operatorname{supp}(f)$ is not compact, just like the one-variable case. However, in favor of the more powerful Lebesgue theory, we will not pursue improper Riemann integrals in this course.