Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

TD05

Exercise numbers agree with Chapter 2 of the lecture notes. References to definitions, propositions, remarks, and theorems refer to the lecture notes.

2.2 Change of variables

2.2.1. Prove Lemma 2.3.

2.2.2. Prove Proposition 2.5.

2.2.3. Prove Proposition 2.18 and show that the functional $\int_U$ there is *not* strictly positive.
Hint: consider the characteristic function of a single point.

2.2.4. Prove Theorem 2.19.

2.2.5. (**Basics on Jordan sets, Jordan contents and more on Riemann integration**) A subset $A \subseteq \mathbb{R}^n$ is called **Jordan measurable**, or **contented**, or simply a **Jordan set**, if its characteristic function $\xi_A : \mathbb{R}^n \to \mathbb{R}$ is Riemann integrable in the sense of Definition 1.5. The collection of all Jordan sets in $\mathbb{R}^n$ is denoted by $\mathcal{J}(\mathbb{R}^n)$. Given an $A \in \mathcal{J}(\mathbb{R}^n)$, the number $v(A) \coloneqq \int_{\mathbb{R}^n} \xi_A$ is called the **Jordan content** of A, and is often interpreted as the (n-dimensional) volume of A. A Jordan set with Jordan content zero is called **Jordan negligible**.

(1) Show that for any $A, B \in \mathcal{J}(\mathbb{R}^n)$, we have $A \cup B, A \cap B, A \setminus B \in \mathcal{J}(\mathbb{R}^n)$.
Hint: consider $\xi_A \wedge \xi_B$, and linear combinations $\xi_A + \xi_B - \xi_A \wedge \xi_B$ and $\xi_A - \xi_A \wedge \xi_B$.

(2) Show that for each $\sigma \in S_n$ and each $A \subseteq \mathbb{R}^n$, we have $A \in \mathcal{J}(\mathbb{R}^n)$ if and only if $T_\sigma A \in \mathcal{J}(\mathbb{R}^n)$, in which case $v(A) = v(T_\sigma(A))$.
Hint: Fubini's theorem.

(3) More generally, using the change of variables formula in Theorem 2.19, show that for each $\varphi \in \mathrm{GL}(n, \mathbb{R})$, and $A \subseteq \mathbb{R}^n$, we have

$$A \in \mathcal{J}(\mathbb{R}^n) \iff \varphi(A) \in \mathcal{J}(\mathbb{R}^n),$$

in which case $v(\varphi(A)) = |\det \varphi| v(A)$.

(4) Show that for any linear $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ and $A \in \mathcal{J}(\mathbb{R}^n)$, we still have $\varphi(A) \in \mathcal{J}(\mathbb{R}^n)$ and $v(\varphi(A)) = |\det \varphi| v(A)$.

(5) Show that $\emptyset \in \mathcal{J}(\mathbb{R}^n)$ with $v(\emptyset) = 0$, but any unbounded subset of $\mathbb{R}^n$ cannot be Jordan measurable.

(6) Show that if $A, B \in \mathcal{J}(\mathbb{R}^n)$, then $v(A \cup B) \leq v(A) + v(B)$, with equality holding if and only if $A \cap B$ is Jordan negligible.

(7) Give an example of a bounded set that is *not* Jordan measurable.

Hint: consider the characteristic function of rational numbers in a bounded open set.

(8) Show that if $A \in \mathcal{J}(\mathbb{R}^n)$, then $\mathrm{Int}(A), \overline{A} \in \mathcal{J}(\mathbb{R}^n)$ with

$$v(A) = v(\mathrm{Int}(A)) = v(\overline{A}).$$

Hint: Find, using the Riemann integrability, an increasing (resp. decreasing) sequence (g_n) (resp. (h_n)) in $C_c(\mathbb{R}^n)$ that converges pointwise to $\xi_{\mathrm{Int}(A)}$ (resp. $\xi_{\overline{A}}$).

(9) Show that for any $A \subseteq \mathbb{R}^n$, we have $A \in \mathcal{J}(\mathbb{R}^n)$ if and only if A is bounded with its boundary ∂A being Jordan negligible.

(10) Show that each bounded interval in $\mathbb{R}$ is Jordan measurable in $\mathbb{R}$ and compute its Jordan content.

Hint: for any $]a, b[$ with $a, b \in \mathbb{R}$ and $a < b$, and for any $\varepsilon > 0$, find $g, h \in C_c(\mathbb{R})$ such that $g \le \xi_{]a,b[} \le h$ and $\int_{\mathbb{R}} (h - g) < \varepsilon$.

(11) Show that every bounded rectangular open set in $\mathbb{R}^n$ is Jordan measurable and compute its Jordan content in terms of its side lengths.

Hint: let $U =]a_1, b_1[\times \cdots \times]a_n, b_n[$ be such a set. Choose $g_i, h_i \in C_c(\mathbb{R})$ with $0 \le g_i \le \xi_{]a_i,b_i[} \le h_i$ for each i. Show that the choice can be made so that $\int_{\mathbb{R}^n} (h - g)$ is as small as we want, where $g(x) = g_1(x_1) \cdots g_n(x_n)$ and $h(x) = h_1(x_1) \cdots h_n(x_n)$.

(12) Show that $A \subseteq \mathbb{R}^n$ is Jordan negligible if and only if for each $\varepsilon > 0$, there is a cover of A by finitely many bounded rectangular open sets such that the sum of all of their Jordan contents does not exceed ε.

(13) Show that finite unions and subsets of Jordan negligible sets remain Jordan negligible.

(14) Show that countable unions of Jordan negligible sets need not be Jordan measurable.

(15) Show that if $A \in \mathcal{J}(\mathbb{R}^{n-1})$ and $f : \overline{A} \to \mathbb{R}$ is continuous, then the set

$$\mathrm{graph}(f|_A) = \left\{ (x, f(x)) \in \mathbb{R}^{n-1} \times \mathbb{R}^1 = \mathbb{R}^n \;\middle|\; x \in A \right\}$$

is Jordan negligible in $\mathbb{R}^n$.

Hint: explain why it suffices to treat the case where A is closed. Exploit the uniform continuity of f, and cover $\mathrm{graph}(f)$ with finitely many small rectangular open sets.

(16) Show that in $\mathbb{R}^n$, all bounded closed balls and n-simplices are Jordan measurable. Here, the word "n-simplices" is the plural form of "n-simplex", which in turn is defined as a set of the form

$$\Delta(v_1, \ldots, v_n) \coloneqq \left\{ \sum_{i=1}^n t_i v_i \;\middle|\; \sum_{i=1}^n t_i \le 1, \text{ and each } t_i \ge 0 \right\}, \qquad v_1, \ldots, v_n \in \mathbb{R}^n.$$

Hint: Consider the boundaries, induct on n, and show that in each case, the boundary is a finite union of Jordan sets, which, up to permutation, are all graphs of continuous functions defined on closed Jordan sets in $\mathbb{R}^{n-1}$.

Background: simplices are often considered as "higher-dimensional triangles". To understand this phrase, draw some simplices in dimension 2 and perhaps dimension 3.

(17) Show that for each $A \in \mathcal{J}(\mathbb{R}^n)$, we have

$$v(A) = \overline{\int_{\mathbb{R}}} \cdots \overline{\int_{\mathbb{R}}} \xi_A(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n = \underline{\int_{\mathbb{R}}} \cdots \underline{\int_{\mathbb{R}}} \xi_A(x_1, \ldots, x_n) \mathrm{d}x_1 \cdots \mathrm{d}x_n.$$

(18) Give a formula for the Jordan content of an n-simplex.

(19) Show that for each $A \in \mathcal{J}(\mathbb{R}^n)$, if $f : \mathbb{R}^n \to \mathbb{R}$ is Riemann integrable, then so is $\xi_A f$. We define $\int_A f$ to be $\int_{\mathbb{R}^n} \xi_A f$ in the following.

(20) Suppose $A \in \mathcal{J}(\mathbb{R}^n)$, and $f : A \to \mathbb{R}$ is a function. We say f is Riemann integrable if $\widetilde{f} : \mathbb{R}^n \to \mathbb{R}$ is Riemann integrable, where $\widetilde{f}(\mathbb{R}^n \setminus A) = 0$ and $\widetilde{f}\Big|_A = f$. In this case, we define $\int_A f$ to be $\int_{\mathbb{R}^n} \widetilde{f}$. Let $\mathcal{R}(A)$ be the space of Riemann integrable functions from A to $\mathbb{R}$. Show that

(20.a) $\mathcal{R}(A)$ is a linear subspace of the vector space of all functions on A;

(20.b) $f, g \in \mathcal{R}(A)$ implies $f \vee g, f \wedge g \in \mathcal{R}(A)$;

(20.c) $f \in \mathcal{R}(A)$ implies $f \wedge 1 \in \mathcal{R}(A)$;

(20.d) $\int_A : \mathcal{R}(A) \to \mathbb{R}$ is a positive linear functional.

(21) Given $A, B, U, V \in \mathcal{J}(\mathbb{R}^n)$ with U, V being open, $U \subseteq A$, $V \subseteq B$, and $\varphi : A \to B$ a C^1-map that restricts to a C^1-diffeomorphism $\varphi|_U : U \to V$ onto V, show that if $A \setminus U$ and $B \setminus V$ are Jordan negligible, and $\det \varphi'$ is bounded on A, then for each $f \in \mathcal{R}(B)$, we have $(f \circ \varphi)|\det \varphi'| \in \mathcal{R}(A)$ and

$$\int_B f = \int_A (f \circ \varphi)|\det \varphi'|.$$

(22) Let $n = 2$. Apply the previous question to compute the Jordan content (aka. the area in classical terms) of $B(a, r) \subseteq \mathbb{R}^2$ with arbitrary $a \in \mathbb{R}^2$ and $r \in]0, +\infty[$.

Hint: use polar coordinates $\varphi(r, \theta) = (r \cos\theta, r \sin\theta)$.

(23) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, $(x, y) \mapsto e^{-x^2-y^2}$. Show that for each $r > 0$, we have

$$\int_{B(0,r)} f = 2\pi \int_0^r e^{-s^2} s\,\mathrm{d}s = \pi\left(1 - e^{-r^2}\right).$$

(24) For $r > 0$, set $C(0, r) \coloneqq \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \|(x, y)\|_\infty \leq r\right\}$. Show that

$$B(0, r) \subseteq C(0, r) \subseteq B(0, 2r).$$

(25) Show that $\int_{\mathbb{R}} e^{-t^2}\mathrm{d}t = \sqrt{\pi}$.

Here the integral is understood as an improper Riemann integral.

Remark. Most of this exercise will be superseded by the more powerful theory of integration in the next chapter. Nevertheless, by wrestling with Riemann theory as presented in this exercise, one will appreciate much more the theory to be presented in the next chapter.

2.3 Riemann integration as a functional

2.3.1. Prove Proposition 3.1.

2.3.2. Assume $n \in \mathbb{N}^+$ in the following.

(1) Show that for each $f \in C_c(\mathbb{R}^n)$, we have $\int f \leq \int |f|$.

(2) Show that $\|\cdot\|_1 : C_c(\mathbb{R}^n) \to \mathbb{R}$, $f \mapsto \int |f|$ is a norm on $C_c(\mathbb{R}^n)$.

(3) Show that $\int_{\mathbb{R}^n} : \left(C_c(\mathbb{R}^n), \|\cdot\|_1\right) \to \mathbb{R}$ is continuous.

(4) Show that the normed space $\left(C_c(\mathbb{R}^n), \|\cdot\|_1\right)$ is *not* complete.

(5) Show that $\int_{\mathbb{R}^n} : \left(C_c(\mathbb{R}^n), \|\cdot\|_\infty\right) \to \mathbb{R}$ is *not* continuous.

2.3.3. Show that if $(f_m)_{m\geq 1}$ is a monotone sequence in $C_c(\mathbb{R}^n)$ whose pointwise limit f remains in $C_c(\mathbb{R}^n)$, then $\lim_{m\to\infty} \int_{\mathbb{R}^n} f_m = \int_{\mathbb{R}^n} f$.

Hint: use Proposition 3.2.

2.3.4. A **rigid motion** on $\mathbb{R}^n$ is defined to be a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ that is distance-preserving in the sense that $\|\varphi(x) - \varphi(y)\| = \|x - y\|$ (recall that $\|\cdot\|$ is the Euclidean norm) for all $x, y \in \mathbb{R}^n$.

(1) We first analyze the structure of rigid motions.

(1.a) Show that for all $x, y \in \mathbb{R}^n$, we have the following polarization identity:

$$x \cdot y = \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right).$$

(1.b) Show that if a rigid motion $\psi : \mathbb{R}^n \to \mathbb{R}^n$ fixes the origin 0, then it preserves the inner product, i.e. for all $x, y \in \mathbb{R}^n$, we have $x \cdot y = \psi(x) \cdot \psi(y)$.

(1.c) Show that a rigid motion $\psi : \mathbb{R}^n \to \mathbb{R}^n$ fixing the origin sends the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ to an orthonormal basis, and $\psi(x) = \sum_{i=1}^n x_i \psi(e_i)$ for each $x = (x_i) \in \mathbb{R}^n$.

(1.d) Given a rigid motion $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, show that there exist a unique $x_0 \in \mathbb{R}^n$ and a unique $A \in O(n, \mathbb{R})$ (we denote by $O(n, \mathbb{R})$ the group of orthogonal real matrices of order n), such that $\varphi(x) = Ax + x_0$ for each $x \in \mathbb{R}^n$, where we view elements in $\mathbb{R}^n$ as column vectors.

(2) Show that the set $M(\mathbb{R}^n)$ of all rigid motions on $\mathbb{R}^n$ forms a group under composition. In particular, rigid motions are bijective from $\mathbb{R}^n$ to $\mathbb{R}^n$.

(3) Show that $\int_{\mathbb{R}^n} : C_c(\mathbb{R}^n) \to \mathbb{R}$ is invariant under the group of rigid motions, i.e. for each $\varphi \in M(\mathbb{R}^n)$ and each $f \in C_c(\mathbb{R}^n)$, we have $\int_{\mathbb{R}^n} f \circ \varphi = \int_{\mathbb{R}^n} f$.

Cultural background. We may put the above exercise in the much wider context of Riemannian manifolds. Although we won't elaborate on this in the course due to time constraints, we point out that a Riemannian manifold M is just a manifold equipped with a good notion of metric (called the Riemannian metric), so that we may talk about its isometry group G, which consists of distance-preserving bijections on M. It is a remarkable theorem of Myers-Steenrod that this isometry group G is a Lie group if M is connected. One may pick out a distinguished integral on the Riemannian manifold M, known as integration with respect to the volume density. As a linear functional, this integral is G-invariant. The above exercise can be regarded as the prototype example of this more general fact. As expected, the G-invariance is most interesting if the natural action $G \curvearrowright M$ is transitive, so that M possesses enough symmetries, just as in the case outlined in the above exercise. Another interesting fact is the deep Nash isometric embedding theorem, which asserts that any Riemannian manifold has a realization in some $\mathbb{R}^n$ not only as a smooth manifold, but also with a compatible metric at the infinitesimal level. To pursue the continuous/smooth symmetries, the interested readers are encouraged to learn more about analysis on manifolds, Lie groups and Riemannian geometry, for which our course can serve as a sound preparation.