Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

Project II. Zero sets of smooth functions, part I

The purpose of this project is to compare the zero sets of polynomial, real analytic, smooth, and continuous functions on $\mathbb{R}$. In particular, we will show that every closed subset of $\mathbb{R}$ is the zero set of a smooth function. All functions in this project are real-valued unless otherwise stated.

II.1. Show that a subset $A \subseteq \mathbb{R}$ is the zero set of a polynomial function on $\mathbb{R}$ if and only if A is finite (possibly empty) or $A = \mathbb{R}$.

II.2. A function $f : \mathbb{R} \to \mathbb{R}$ is called *real analytic* if it can be locally represented by a convergent power series. Equivalently, $f \in C^\infty(\mathbb{R}, \mathbb{R})$ and, for each $s \in \mathbb{R}$, the Taylor series of f at s converges to f on a neighborhood of s.

Show that the zero set of a nonzero real analytic function on $\mathbb{R}$ is closed and has no accumulation point in $\mathbb{R}$. In particular, for every $t_0 \in f^{-1}(0)$, there is a neighborhood I of t_0 such that $I \cap f^{-1}(0) = \{t_0\}$.

Hint: first consider a power series representing f near an accumulation point of its zeros, then use the connectedness of $\mathbb{R}$.

II.3. Define $f_0(t) = (1 + t^2)^{-1}$.

(1) Show that f_0 is real analytic by giving its power series expansion at each $r \in \mathbb{R}$, and determine the radius of convergence.

Hint: use geometric series and the identity

$$\frac{1}{1 + (r + h)^2} = \frac{1}{2i}\left(\frac{1}{r + h - i} - \frac{1}{r + h + i}\right), \qquad i^2 = -1.$$

(2) Can f_0 be represented globally by a power series $\sum_{j \geq 0} a_j t^j$ that converges to $f_0(t)$ for every $t \in \mathbb{R}$?

(3) What is the image of f_0?

(4) Suppose a power series $g(t) = \sum_{j \geq 0} b_j t^j$ converges for every $t \in \mathbb{R}$. Must g be bounded on $\mathbb{R}$?

Hint: compare $g(t) = \sin(t)$ and $g(t) = t$.

Background: in complex analysis, Liouville's theorem says that a power series $h(z) = \sum_{j \geq 0} c_j z^j$ converging everywhere on $\mathbb{C}$ and bounded on $\mathbb{C}$ must be constant.

II.4. Suppose $A \subseteq \mathbb{R}$ is closed and has no accumulation point in $\mathbb{R}$. Show that A is countable.

II.5. Establish the converse of Exercise II.2: every closed subset $A \subseteq \mathbb{R}$ without accumulation points is the zero set of a real analytic function on $\mathbb{R}$.

(1) Explain why it suffices to treat the case where A is infinite.

(2) Suppose A is infinite. Show that there exists a strictly increasing sequence $(c_j)_{j \geq 0}$ of positive real numbers such that

$$\lim_{j \to \infty} c_j = \infty, \qquad \left\{ \pm c_j \;\middle|\; j \geq 0 \right\} \cap A = \emptyset.$$

(3) Show that $A_j \coloneqq A \cap \,]-c_j, c_j[$ is finite for every $j \geq 0$.

Hint: $A_j = A \cap [-c_j, c_j]$.

(4) Set $J \coloneqq \left\{ j \in \mathbb{N} \;\middle|\; A_{j+1} \setminus A_j \neq \emptyset \right\}$. For a polynomial $p(t) = \sum_{k=0}^{d} a_k t^k$ and $R > 0$, write $\|p\|_R \coloneqq \sum_{k=0}^{d} |a_k| R^k$. Show that, for every $j \in J$, there exists a polynomial p_j such that

$$\|p_j\|_{c_j} \leq 2^{-j}, \qquad p_j^{-1}(1) = A_{j+1} \setminus A_j.$$

In particular, $\left|p_j(t)\right| \leq 2^{-j}$ for $|t| \leq c_j$.

Hint: if $A_{j+1} \setminus A_j = \{r_1, \ldots, r_N\}$, consider

$$p_j(t) = 1 - \prod_{k=1}^{N} \left(1 - \left(\frac{t}{r_k}\right)^M\right)$$

for a sufficiently large odd positive integer M. Use $|r_k| > c_j$.

(5) Let $(a_j)_{j\geq 0}$ be real numbers with $\sum_{j\geq 0} \left|a_j\right| < \infty$. Show that $\prod_{j\geq 0}(1 - a_j)$ converges, and that its value is zero if and only if $a_j = 1$ for some j.

Hint: discard finitely many factors so that $\left|a_j\right| \leq 1/2$, then use the logarithm.

(6) For a power series $u(t) = \sum_{k\geq 0} a_k t^k$ with infinite radius of convergence, set

$$\|u\|_R \coloneqq \sum_{k\geq 0} |a_k| R^k.$$

Let $(u_j)_{j\geq 0}$ be such power series, and suppose that $\sum_{j\geq 0} \|u_j\|_R < \infty$ for every $R > 0$. Show that both

$$\sum_{j\geq 0} u_j(t) \qquad \text{and} \qquad \prod_{j\geq 0} (1 - u_j(t))$$

converge uniformly on each bounded interval and are represented by power series with infinite radius of convergence.

Hint: use absolute convergence to interchange sums. For products, first show that $\|uv\|_R \leq \|u\|_R \|v\|_R$, then expand the product over finite subsets of indices and bound the sum of the resulting norms by $\prod_{j\geq 0}(1 + \|u_j\|_R)$.

(7) Choose p_j as in part (4) for every $j \in J$. Show that there exists a real analytic function f with $f^{-1}(0) = A$ whose Taylor series at every point has infinite radius of convergence.

Hint: take a polynomial q_0 with $q_0^{-1}(0) = A_0$ and consider

$$f = q_0 \prod_{j\in J} (1 - p_j).$$

Apply parts (5) and (6). To expand at another point, use the binomial formula and absolute convergence.

II.6. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = \begin{cases} 0, & t \leq 0, \\ e^{-1/t}, & t > 0. \end{cases}$$

Show that $g \in C^\infty(\mathbb{R}, \mathbb{R})$ and that, for each $k \in \mathbb{N}$, there exists a polynomial $P_k \in \mathbb{R}[X]$ such that

$$g^{(k)}(t) = P_k\left(\frac{1}{t}\right) e^{-1/t} \qquad \text{for } t > 0.$$

II.7. What is the Taylor series of the function g in Exercise II.6 at 0? Is g real analytic? What is its range?

II.8. Show that, for any $a, b \in \mathbb{R}$ with $a < b$, there exists $\varphi_{a,b} \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$\varphi_{a,b}^{-1}(0) =]-\infty, a] \cup [b, \infty[,$$

and $\varphi_{a,b}(t) > 0$ for $a < t < b$.

II.9. Show that every open subset of $\mathbb{R}$ is a countable union of bounded open intervals.

II.10. Let $(f_j)_{j\geq 1}$ be a sequence in $C^\infty(\mathbb{R}, \mathbb{R})$ satisfying the following conditions:

(i) $(f_j(t_0))_{j\geq 1}$ converges at some $t_0 \in \mathbb{R}$;

(ii) for each $k \geq 1$, the sequence $(f_j^{(k)})_{j\geq 1}$ converges uniformly on every compact interval.

Show that f_j converges uniformly on every bounded interval to a smooth function f, and that

$$f^{(k)}(t) = \lim_{j\to\infty} f_j^{(k)}(t) \qquad \text{for all } k \in \mathbb{N} \text{ and } t \in \mathbb{R}.$$

Hint: pass to the limit in

$$f_j(t) = f_j(t_0) + \int_{t_0}^{t} f_j'(s)\,\mathrm{d}s$$

on a compact interval containing t_0, and repeat for higher derivatives. A version in several variables is developed in Project III.

II.11. Let $A \subseteq \mathbb{R}$ be closed and set $U := \mathbb{R} \setminus A$. Treat $U = \emptyset$ separately. Otherwise, write $U = \bigcup_{j\geq 1} I_j$, where the I_j are bounded open intervals, and choose $h_j \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $h_j > 0$ on I_j and $h_j = 0$ outside I_j. Show that there are constants $c_j > 0$ such that $\sum_{j\geq 1} c_j h_j$ converges to a smooth function h with $h^{-1}(0) = A$.

Hint: choose c_j so that

$$c_j \sup_{t\in\mathbb{R}} \left| h_j^{(k)}(t) \right| \leq 2^{-j} \qquad \text{for } 0 \leq k \leq j,$$

and apply Exercise II.10 to the partial sums.

II.12. What are the zero sets of continuous real-valued functions on $\mathbb{R}$? Give a simple proof that, for every closed set $A \subseteq \mathbb{R}$, there exists a continuous function $u : \mathbb{R} \to \mathbb{R}$ such that $u > 0$ on $\mathbb{R} \setminus A$ and $u^{-1}(0) = A$.

Hint: use the distance to A when $A \neq \emptyset$, and treat $A = \emptyset$ separately.