Mathematical Analysis III, 2026 autumn semester
HIT, class of excellence

Project III. Zero sets of smooth functions, part II

Fix $n \in \mathbb{N}^+$. We continue the study of zero sets from Project II, now on $\mathbb{R}^n$. All functions in this project are real-valued. We use $|x|$ for the Euclidean norm and $B(a,r)$ for the open ball of center a and radius r.

III.1. For any $r, s \in \mathbb{R}$ with $r < s$, show that there exists $f \in C^\infty(\mathbb{R}, \mathbb{R})$ taking values in $[0,1]$ such that

$$f(t) = 1 \quad \text{for } t \le r, \qquad f(t) = 0 \quad \text{for } t \ge s.$$

III.2. Let $a \in \mathbb{R}^n$ and $r > 0$. Show that there exists $g \in C^\infty(\mathbb{R}^n, \mathbb{R})$ taking values in $[0,1]$ such that

$$g|_{B(a,r)} = 1, \qquad g|_{\mathbb{R}^n \setminus B(a,2r)} = 0.$$

Hint: apply Exercise III.1 with suitable endpoints and compose the resulting function with $x \mapsto |x-a|^2$.

III.3. Show that any nonempty open set $U \subseteq \mathbb{R}^n$ is a countable union of open balls $B(a_j, r_j)$, $j \in \mathbb{N}^+$, such that $B(a_j, 2r_j) \subseteq U$ for each j.

Hint: consider balls whose radii and center coordinates are rational.

The next three exercises establish the convergence result needed below. Convergence *locally uniformly* on an open set means uniform convergence on each compact subset of that set.

III.4. Let $U \subseteq \mathbb{R}^n$ be open and suppose $f_j \in C^1(U, \mathbb{R})$ for $j \ge 1$. Assume that $\operatorname{grad} f_j$ converges locally uniformly on U. Suppose $\overline{B(a,r)} \subseteq U$, where $r > 0$, and $(f_j(a))_{j\ge1}$ converges. Show that, for all $p, q \ge 1$,

$$\sup_{x \in \overline{B(a,r)}} \bigl|f_p(x) - f_q(x)\bigr| \le \bigl|f_p(a) - f_q(a)\bigr| + r \sup_{x \in \overline{B(a,r)}} \bigl|\operatorname{grad} f_p(x) - \operatorname{grad} f_q(x)\bigr|.$$

Deduce that $(f_j)_{j\ge1}$ converges uniformly on $\overline{B(a,r)}$.

Hint: integrate along the segment joining a to x.

III.5. Let $U \subseteq \mathbb{R}^n$ be connected and open, and suppose $f_j \in C^1(U, \mathbb{R})$, $j \ge 1$. Assume that $\operatorname{grad} f_j$ converges locally uniformly to a map $G : U \to \mathbb{R}^n$, and that $(f_j(a))_{j\ge1}$ converges at some $a \in U$. Show that $(f_j)_{j\ge1}$ converges locally uniformly to a function $f \in C^1(U, \mathbb{R})$ with $\operatorname{grad} f = G$.

Hint: use Exercise III.4 to show that the set of points where $(f_j(x))_{j\ge1}$ converges is both open and closed in U. Then use compactness to obtain uniform convergence on compact subsets. To identify $\partial_i f$, pass to the limit in

$$f_j(x + te_i) - f_j(x) = \int_0^t \partial_i f_j(x + se_i)\,\mathrm{d}s$$

for sufficiently small t, where e_i is the i-th standard basis vector.

III.6. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, write $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Let $U \subseteq \mathbb{R}^n$ be connected and open, and let $f_j \in C^\infty(U, \mathbb{R})$, $j \geq 1$. Suppose that $(f_j(a))_{j\geq 1}$ converges at some $a \in U$ and that $(\partial^\alpha f_j)_{j\geq 1}$ converges locally uniformly for every α with $|\alpha| \geq 1$. Show that f_j converges locally uniformly to a smooth function f, and that

$$\partial^\alpha f = \lim_{j\to\infty} \partial^\alpha f_j \qquad \text{for every } \alpha \in \mathbb{N}^n.$$

Deduce that a series $\sum_{j\geq 1} h_j$ of smooth functions on U has a smooth sum and may be differentiated term by term to every order if, for every compact $K \subseteq U$ and every $\alpha \in \mathbb{N}^n$,

$$\sum_{j\geq 1} \sup_{x\in K} \left|\partial^\alpha h_j(x)\right| < \infty.$$

Hint: apply Exercise III.5 successively to the derivatives, and then apply the result to the partial sums of the series.

III.7. Let $U \subseteq \mathbb{R}^n$ be open. Show that there exists $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ such that $\varphi^{-1}(0) = \mathbb{R}^n \setminus U$.

Hint: treat $U = \emptyset$ separately. Otherwise, write $U = \bigcup_{j\geq 1} B(a_j, r_j)$ as in Exercise III.3. Choose φ_j by Exercise III.2, and choose constants $c_j > 0$ such that

$$c_j \sup_{x\in\mathbb{R}^n} \left|\partial^\alpha \varphi_j(x)\right| \leq 2^{-j} \qquad \text{whenever } |\alpha| \leq j.$$

Apply Exercise III.6 to $\sum_{j\geq 1} c_j \varphi_j$.

III.8. Show that a subset $A \subseteq \mathbb{R}^n$ is closed if and only if there exists $\psi \in C^\infty(\mathbb{R}^n, \mathbb{R})$ with $\psi^{-1}(0) = A$. Explain how this implies the analogous statement for continuous functions. Can you find an easier proof of the latter statement?

III.9. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function. If $p^{-1}(0)$ has nonempty interior, show that $p = 0$.

Hint: consider mixed derivatives.

III.10. A real-valued function on $\mathbb{R}^n$ is called *real analytic* if it is locally represented by a convergent power series in n variables. Generalize Exercise III.9 to real analytic functions.

III.11. Give an example of a closed subset of $\mathbb{R}^n$ that is the zero set of a smooth function but is not the zero set of any real analytic function.

III.12. (Very challenging!) Can you characterize the zero sets of polynomial functions on $\mathbb{R}^n$, $n \geq 2$, as in the one-dimensional case? What about real analytic functions?

Background: the polynomial case belongs to algebraic geometry, while the analytic case belongs to analytic geometry.